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A
TREATISE
ON
FLUXIONS.

BY THE

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A
TREATISE ON FLUXIONS.

SECTION I.

DEFINITIONS.

ART. 1. EVERY quantity is here considered as generated by motion; a line by the motion of a point; a surface by the motion of a line; a solid by the motion of a surface; an angle by the rotation of a line about one of its extremities*.

(2.) The quantity thus generated is called the *fluent*, or *flowing* quantity.

(3.) The *velocities* with which flowing quantities increase or decrease at any point of time, are called the *fluxions* of those quantities at that instant.

COR. 1. As the velocities are in proportion to the increments or decrements which would be generated in a given time, if at any instant the velocities were to become *uniform*, such increments or decrements will represent the fluxions at that instant†.

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* SIR I. NEWTON, in the introduction to his *Quadrature of Curves*, observes that " these geneses really take place in the nature of things, and are daily seen in the motion of bodies. And after this manner, the ancients, by drawing moveable right lines along immoveable right lines, taught the genesis of rectangles."

† This is agreeable to SIR I. NEWTON's ideas on the subject. He says, " I sought a method of determining quantities from the velocities of the motions or increments with which they are generated; and calling these velocities of the motions or increments, *fluxions*, and the generated quantities *fluents*, I fell by degrees upon the method of fluxions."—Introd. to *Quad. Curves*.

Cor. 2. Hence, as any given time may be assumed, the fluxion is not an *absolute* but a *relative* quantity. When we have several cotemporary fluxions, we may assume one fluxion what we please, and thence determine the values of the others. Thus, if x and y increase uniformly, and if x increase by p in the time that y increases by q, then the cotemporary increments of x and y will be p and q, $2p$ and $2q$, $3p$ and $3q$, &c. hence, if p be assumed the fluxion of x, the fluxion of y will be q; if the former fluxion be $2p$, the latter will be $2q$, &c. &c.

Cor. 3. A *constant* quantity has no fluxion.

(4.) The first letters, a, b, c, &c. of the alphabet are usually put for constant quantities, and the last, v, w, x, y, z, for variable ones; and they are to be thus understood, unless the contrary be expressed.

(5.) The fluxion of a simple quantity, as x, is expressed by placing a point over it, thus $\dot{x}$.*

To find the FLUXIONS of QUANTITIES.

Prop. I.

If two quantities increase or decrease uniformly, the increments or decrements generated in a given time, will be as their fluxions.

(6.) This appears from Art. 3. Cor. 1.

Prop. II.

If one quantity increase uniformly, and another of the same kind increase with an accelerated or retarded

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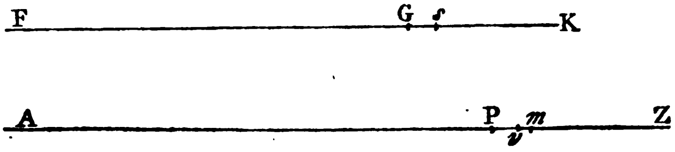
* Foreign mathematicians denote the fluxion of x by dx, which is liable to two objections: first, it is not so simple as $\dot{x}$, and becomes still more complex for the higher orders of fluxions; secondly, dx is a notation which also signifies the product of d multiplied by x. Every notation should have but one meaning.

velocity, and two increments be assumed which are generated in the same time; if those increments be diminished till they vanish, that ratio to which they approach as their limit, is the ratio of the fluxions of those quantities.

(7.) Let the line FK be described with an uniform velocity, and AZ with an accelerated velocity, and let the increments Gs, Pm be generated in the same time; let also Pv be the increment that would have been generated in the same time, if the velocity at P had been continued uniform; then by Prop. 1. the fluxions of FK, AZ, at the points G and P, will be represented by Gs and Pv. Let V be the velocity at P, or the velocity with which Pv is described, and let r be the increase of velocity from P to m; then the velocity at m will be $V+r$, and vm is the increment which is described in consequence of the increase r of velocity since the describing point left P. Now let $V+w$ be the *uniform* velocity with which Pm would be described in the same time that Pv and Pm are described, as before mentioned; then it is manifest, that this uniform velocity must be between the velocities at P and m, that is, $V+w$ is greater than V and less than $V+r$, or w is greater than o and less than r. Also, since the spaces described in the same time are as the velocities, $V : V+w :: Pv : Pm$*. Now



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* If we diminish the times in which these increments are described; then as the points v and m approach to P, Pv will continue to be described with the uniform velocity V; but r will be diminished, and by diminishing the time till it becomes indefinitely small, r will become indefinitely small; but vm is described in consequence of this increase r of velocity; hence, when r becomes indefinitely small with respect to V, the space vm must become indefinitely small in respect to Pv: therefore the ratio of $Pv : Pm$

in *every* state of these increments, $V : V+w :: Pv : Pm$; and by continually diminishing the time, and consequently the increments, we diminish r and w, but V remains constant; it is manifest therefore that the ratio of $V : V+w$, and consequently that of $Pv : Pm$, continually approaches towards a ratio of equality, agreeably to what is shown in the note; and when the time, and consequently the increments, become actually $=0$, then $r=0$; consequently $w=0$; therefore the *limit* of the ratio of $Pv : Pm$ becomes that of $V : V$, a ratio of equality*. Hence, the *limit* of the ratio of $Gs : Pm$ is the same as the *limit* of the ratio of $Gs : Pv$, or it is $Gs : Pv$, that ratio being constant; that is, *the limiting ratio of the increments is the ratio of the fluxions.*

The same is manifestly true for the limiting ratio of the decrements of two quantities; for, conceiving the describing points to move backwards, the decrements sG, mP in this case become the same as the increments in the other; consequently their *limiting* ratio will express the ratio of the fluxions at G and P, or the rate at which FG, AP are, at that instant, decreasing. As the points P, G, may be taken at A, F, respectively, the limiting ratio of two nascent or evanescent quantities, will be the ratio of their fluxions at the instant they begin or cease to be.†

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is, in that state, indefinitely near to a ratio of equality; but it is manifest that it never can become accurately a ratio of equality, because vm will not vanish until Pv and Pm vanish; consequently the ratio of the actual increments $Gs : Pm$ can never accurately express the ratio of the fluxions, that ratio being expressed by the ratio of $Gs : Pv$. We are therefore to consider, to what ratio $Pv : Pm$ approaches as it's *limit*, when we make the time in which the increments are described, and consequently the increments themselves, vanish.

* By keeping the ratio of the vanishing quantities thus expressed by finite quantities, it removes the obscurity which may arise when we consider the quantities themselves; this is agreeable to the reasoning of SIR I. NEWTON in his *Principia*, Lib. I. Sect. 1. Lem. 7, 8, 9.

† Hence, if FG, AP, represent the numerator and denominator

Hence, the *limiting* ratio of the increments or decrements of two quantities which are *both* generated by variable velocities, will be the ratio of their fluxions. And as the velocities with which these two lines increase or decrease, may be made to agree with the rate of increase or decrease of any two quantities which may be compared together, the proposition must be true for quantities of any kind.

COR. As the *limiting* ratio of the increments is the ratio of the fluxions, it is manifest that when the increments are in an increasing or decreasing state, the fluxions will be increasing or decreasing.

(8.) It has been said, that when the increments are actually vanished, it is absurd to talk of any ratio between them. It is true; but we speak not here of any ratio then existing between the quantities, but of that ratio to which they have approached as their *limit*; and that ratio still remains. Thus, let the increments of two quantities be denoted by ax^2+mx and bx^2+nx; then the *limit* of their ratio, when $x=0$, is $m:n$; for in every state of these quantities; $ax^2+mx : bx^2+nx :: ax+m : bx+n ::$ (when $x=0$) $m : n$. As the quantities therefore approach to nothing, the ratio approaches to that of $m : n$ as it's *limit*. Hence, if $m=n$, the *limit* of this ratio is a ratio of equality. We must therefore be careful to distinguish between the ratio of two evanescent quantities, and the *limit* of their ratio; the former ratio

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of a fraction, and by the motion of G and P to A and F respectively, AP, FG, vanish together, the limit of $\frac{AP}{FG}$ is expressed by $\frac{\dot{\overline{AP}}}{\dot{\overline{FG}}}$ at the time when AP, FG vanish. This is what we always mean by the value of $\frac{0}{0}$; which expression has been objected to, because not properly understood.

Ex. The value of $\frac{x^2-a^2}{x-a}$ when $x=a$, is, (taking the fluxions by Prop. 5.) $\frac{2x\dot{x}}{\dot{x}}=2x=2a$.

never arriving at the latter, as the quantities vanish at the instant that such a circumstance is about to take place.

PROP. III.

If the fluxion of x *be denoted by* $\dot{x}$, *the fluxion of* ax *will be* a$\dot{x}$.

(9.) For if x increase uniformly, ax will also increase uniformly, and a times as fast; hence, by Prop. 1. the fluxion of the latter will be a times that of the former, or it will be $a\dot{x}$.

COR. Hence, in taking the fluxion of a variable quantity multiplied into a constant one, the constant multiplier is retained.

PROP. IV.

The fluxion of $x \pm a$ *is* $\dot{x}$.

(10.) For a being constant, and only connected to x by the signs + or −, it does not affect the *increase* or *decrease* of the quantity; therefore the fluxion is the same as the fluxion of x, or it is $\dot{x}$.

COR. Hence, constant quantities connected to variable ones by the signs + or −, disappear when the fluxions are taken. Thus it appears, that the fluxion has no necessary relation to the *magnitude* of the fluent.

PROP. V.

Given ($\dot{x}$) *the fluxion of* x, *to find the fluxion of* x^n, n *being a whole number.*

(11.) Let x increase uniformly by v and become $x+v$, then will x^n become $\overline{x+v}|^n$; but (*Wood's Alg.* Art. 232.) $\overline{x+v}|^n = x^n + nx^{n-1}v + n.\frac{n-1}{2}x^{n-2}v^2 + \&c.$

and if from this quantity we take x^n, there remains $nx^{n-1}v+n.\frac{n-1}{2}x^{n-2}v^2+$ &c. for the cotemporary increment of x^n; but although x increases uniformly by v, x^n does not increase uniformly; for if in the increment of x^n we substitute 1, 2, 3, &c. for v, and take the differences of the results, these differences will not be equal; hence, to get the ratio of the fluxion of x to the fluxion of x^n we must, according to Prop. 2. take the *limiting* ratio of the increments. Now the increment of x : the increment of x^n :: $v : nx^{n-1}v+n.\frac{n-1}{2}x^{n-2}v^2+$ &c. :: $1 : nx^{n-1}+n.\frac{n-1}{2}x^{n-2}v+$ &c. and to get the *limiting* ratio of these increments, we must make $v=0$, in which case the ratio becomes $1 : nx^{n-1}$, which therefore expresses the ratio of the fluxion of x to the fluxion of x^n; but $\dot{x}$ denotes the fluxion of x, therefore $nx^{n-1}\dot{x}$ represents the cotemporary fluxion of x^n.

If $n=0$, $x^n=1$ a constant quantity; therefore by Art. 3. Cor. 3. it has no fluxion.

Prop. VI.

To find the fluxion of $x^{\frac{n}{m}}$, m *and* n *being any whole numbers.*

(12.) Put $y=x^{\frac{n}{m}}$, then $y^m=x^n$; hence, by taking the fluxions, $my^{m-1}\dot{y}=nx^{n-1}\dot{x}$, $\therefore\ \dot{y}=\frac{nx^{n-1}\dot{x}}{my^{m-1}}$ = (by substituting for y it's value in terms of x) $\frac{nx^{n-1}\dot{x}}{mx^{\frac{nm-n}{m}}}=$ $\frac{nx^{n-1}\dot{x}}{mx^{n-\frac{n}{m}}}=\frac{n}{m}\times x^{\frac{n}{m}-1}.\dot{x}$

Cor. Let the root be a compound quantity as

a^m+x^m, to find the fluxion of $\overline{a^m+x^m}|^{\frac{1}{n}}$. Put $y = \overline{a^m+x^m}|^{\frac{1}{n}}$, then $y^n = a^m+x^m$, and $ny^{n-1}\dot{y} = mx^{m-1}\dot{x}$; hence, $\dot{y} = \frac{mx^{m-1}\dot{x}}{ny^{n-1}} = \frac{mx^{m-1}\dot{x}}{n\times\overline{a^m+x^m}|^{\frac{n-1}{n}}} = \frac{1}{n}\times\overline{a^m+x^m}|^{\frac{1-n}{n}}\times mx^{m-1}\dot{x} = \frac{1}{n}\times\overline{a^m+x^m}|^{\frac{1}{n}-1}\times mx^{m-1}\dot{x}$.

(13.) Hence it appears, that whether the root be a simple or a compound quantity, the fluxion of any power thereof is found by the following

RULE.

Multiply by the index, diminish the index by unity, and multiply by the fluxion of the root.

EXAMPLES.

Ex. 1. The fluxion of x^9 is $9x^8\dot{x}$.

Ex. 2. The fluxion of $3y^5$ is $15y^4\dot{y}$.

Ex. 3. The fluxion of $\frac{3}{2}y^{\frac{4}{7}}$ is $\frac{12}{14}y^{-\frac{3}{7}}\dot{y} = \frac{6\dot{y}}{7y^{\frac{3}{7}}}$.

Ex. 4. The fluxion of $\frac{5}{9}x^{\frac{7}{11}}$ is $\frac{35}{99}x^{-\frac{4}{11}}\dot{x} = \frac{35\dot{x}}{99x^{\frac{4}{11}}}$.

Ex. 5. The fluxion of $\frac{4}{7}x^{\frac{11}{9}}$ is $\frac{44}{63}x^{\frac{2}{9}}\dot{x}$.

Ex. 6. What is the fluxion of $\overline{a^2+x^2}|^3$?

Here the root is a^2+x^2, and it's fluxion $2x\dot{x}$; hence, the fluxion required is $3\times\overline{a^2+x^2}|^2\times 2x\dot{x} = \overline{a^2+x^2}|\times 6x\dot{x}$.

Ex. 7. What is the fluxion of $\sqrt{a^2+x^2}$, or of $\overline{a^2+x^2}|^{\frac{1}{2}}$?

Here the root is a^2+x^2, and it's fluxion $2x\dot{x}$; hence, the fluxion is $\frac{1}{2}\times\overline{a^2+x^2}|^{-\frac{1}{2}}\times 2x\dot{x} = \frac{x\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}}$.

Ex. 8. What is the fluxion of $\overline{x^2+y^2}\big|^{\frac{3}{2}}$?

Here the root is x^2+y^2, and it's fluxion $2x\dot{x}+2y\dot{y}$; hence, the fluxion required is $\frac{3}{2}\times\overline{x^2+y^2}\big|^{\frac{1}{2}}\times(2x\dot{x}+2y\dot{y})$ $=3\times\overline{x^2+y^2}\big|^{\frac{1}{2}}\times(x\dot{x}+y\dot{y})$.

Ex. 9. What is the fluxion of $\overline{x+y}\big|^2$?

Here the root is $x+y$, and it's fluxion $\dot{x}+\dot{y}$; hence, the fluxion required is $2\times(x+y)\times(\dot{x}+\dot{y})$.

Ex. 10. What is the fluxion of $\overline{a^5+x^5}\big|^{\frac{1}{2}}$?

Here the root is a^5+x^5, and it's fluxion $5x^4\dot{x}$; hence, the fluxion required is $\frac{1}{2}\times\overline{a^5+x^5}\big|^{-\frac{1}{2}}\times 5x^4\dot{x}=$

$$\frac{5x^4\dot{x}}{2\times\overline{a^5+x^5}\big|^{\frac{1}{2}}}.$$

Ex. 11. What is the fluxion of $\dfrac{1}{\overline{a^2+x^2}\big|^{\frac{5}{9}}}$?

This quantity becomes $\overline{a^2+x^2}\big|^{-\frac{5}{9}}$, and the root is a^2+x^2, whose fluxion is $2x\dot{x}$; hence, the fluxion required is $-\frac{5}{9}\times\overline{a^2+x^2}\big|^{-\frac{14}{9}}\times 2x\dot{x}=\dfrac{-10x\dot{x}}{9\times\overline{a^2+x^2}\big|^{\frac{14}{9}}}$. In like manner, bring any quantity from the denominator up to the numerator, by changing the sign of the index, and then proceed by the rule.

Ex. 12. What is the fluxion of $\overline{ax^2+by^3+cz^4}\big|^{\frac{7}{3}}$?

Here the root is $ax^2+by^3+cz^4$, and its fluxion $2ax\dot{x}+3by^2\dot{y}+4cz^3\dot{z}$; hence, the fluxion required is $\frac{7}{3}\times\overline{ax^2+by^3+cz^4}\big|^{\frac{4}{3}}\times(2ax\dot{x}+3by^2\dot{y}+4cz^3\dot{z})$.

Ex. 13. What is the fluxion of $\sqrt{x^2+\sqrt{a^2+y^2}}$?

Put $z=\sqrt{x^2+\sqrt{a^2+y^2}}$, then $z^2=x^2+\sqrt{a^2+y^2}$; now the fluxion of $\sqrt{a^2+y^2}$, or of $\overline{a^2+y^2}\big|^{\frac{1}{2}}$, is $\frac{1}{2}\times$

$\overline{a^2+y^2}|^{-\frac{1}{2}} \times 2y\dot{y} = \overline{a^2+y^2}|^{-\frac{1}{2}} \times y\dot{y}$; hence, $2z\dot{z} = 2x\dot{x} + \overline{a^2+y^2}|^{-\frac{1}{2}} \times y\dot{y}$, therefore $\dot{z} = \frac{2x\dot{x} + \overline{a^2+y^2}|^{-\frac{1}{2}} \times y\dot{y}}{2z} = \frac{2x\dot{x} + \overline{a^2+y^2}|^{-\frac{1}{2}} \times y\dot{y}}{2\sqrt{x^2+\sqrt{a^2+y^2}}}$.

Prop. VII.

To find the fluxion of a product x y.

(14.) The fluxion of $(x+y)^2$, by the last rule, is $2 \times (x+y) \times (\dot{x}+\dot{y}) = 2x\dot{x} + 2x\dot{y} + 2y\dot{x} + 2y\dot{y}$; also, $(x+y)^2 = x^2 + 2xy + y^2$, whose fluxion is $2x\dot{x}$ + *the fluxion of* $2xy + 2y\dot{y}$; make these two values of the fluxion of $(x+y)^2$ equal to each other, omit the first and last terms which are common to both, and we have *the fluxion of* $2xy = 2x\dot{y} + 2y\dot{x}$; hence, the fluxion of xy is $x\dot{y} + y\dot{x}$.

Otherwise thus. If we suppose x constant, the fluxion of xy is $x\dot{y}$ by Prop. 3; and if we suppose y constant, the fluxion is $y\dot{x}$; hence, if neither be constant, the fluxion is $x\dot{y} + y\dot{x}$.

Cor. Hence, we may find the fluxion of xyz. For if $v = xyz$, and $w = xy$, then $v = wz$, and $\dot{v} = w\dot{z} + z\dot{w}$; but $w = xy$, $\therefore \dot{w} = x\dot{y} + y\dot{x}$; substitute these values for w and $\dot{w}$ and we get $\dot{v} = xy\dot{z} + zx\dot{y} + zy\dot{x}$.

(15.) In like manner we proceed for any number of factors; hence, the fluxion of the product of any number of quantities is found by the following

RULE.

Multiply the fluxion of each quantity into the product of all the rest, and the sum of all the products is the fluxion required.

EXAMPLES.

Ex. 1. The fluxion of x^2y^3 is $x^2 \times 3y^2\dot{y} + y^3 \times 2x\dot{x} = 3x^2y^2\dot{y} + 2y^3x\dot{x}$.

Ex. 2. The fluxion of $y^{\frac{7}{2}}x^{\frac{5}{3}}z$ is $x^{\frac{5}{3}}z\times\frac{7}{2}y^{\frac{5}{2}}\dot{y}+y^{\frac{7}{2}}z\times\frac{5}{3}x^{\frac{2}{3}}\dot{x}+y^{\frac{7}{2}}x^{\frac{5}{3}}\dot{z}=\frac{7}{2}x^{\frac{5}{3}}zy^{\frac{5}{2}}\dot{y}+\frac{5}{3}y^{\frac{7}{2}}zx^{\frac{2}{3}}\dot{x}+y^{\frac{7}{2}}x^{\frac{5}{3}}\dot{z}$.

Ex. 3. The fluxion of $w^mx^ny^rz^s$ is $mx^ny^rz^sw^{m-1}\dot{w}+nw^my^rz^sx^{n-1}\dot{x}+rw^mx^nz^sy^{r-1}\dot{y}+sw^mx^ny^rz^{s-1}\dot{z}$.

Ex. 4. To find the fluxion of $x^2\times\overline{a^4+y^4}|^{\frac{3}{2}}$.

By the last rule, the fluxion of $\overline{a^4+y^4}|^{\frac{3}{2}}$ is $\frac{3}{2}\times\overline{a^4+y^4}|^{\frac{1}{2}}\times 4y^3\dot{y}=6\times\overline{a^4+y^4}|^{\frac{1}{2}}\times y^3\dot{y}$; hence, the fluxion required is $x^2\times 6\times\overline{a^4+y^4}|^{\frac{1}{2}}\times y^3\dot{y}+\overline{a^4+y^4}|^{\frac{3}{2}}\times 2x\dot{x}$.

Ex. 5. To find the fluxion of $\sqrt{a^2+x^2}\times\sqrt{b^2+y^2}$. Find the fluxion of each part by the last rule, and the fluxion required is $\sqrt{a^2+x^2}\times\frac{y\dot{y}}{\sqrt{b^2+y^2}}+\sqrt{b^2+y^2}\times\frac{x\dot{x}}{\sqrt{a^2+x^2}}$.

(16.) It appears from this Prop. that the fluxion of xy consists of two parts, $x\dot{y}$, and $y\dot{x}$, the former part arising from the increase of y by $\dot{y}$, and the latter from the increase of x by $\dot{x}$; but if x should decrease whilst y increases, then the fluxion, expressing the *increase* of xy upon the whole, will be $x\dot{y}-y\dot{x}$, being the increase *minus* the decrease. Hence, to express the rate at which any quantity *increases*, the fluxion of the parts which increase must be written with the sign +, and those which decrease with the sign −*. Now the increasing quantity is considered as positive; but if a negative quantity increase in magnitude, it must be considered as a decreasing quantity, and it's fluxion will be negative. In like manner, a negative quantity decreasing in magnitude must be considered as an increasing quantity, and it's fluxion will be positive. If therefore the fluxions of increasing quantities be written with the sign $+$, and of decreasing with $-$, whenever the fluxion of any quantity is positive, it shows that quantity to be in an increasing state; and when negative, to be in a decreasing state. In like manner, if $x^2+y^2=$ a constant quantity, then if x decrease and y increase, the fluxion is $-2x\dot{x}+2y\dot{y}=0$.

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* Hence it appears, that when a quantity passes through a maximum or minimum, the fluxion on each side has a different sign.

PROP. VIII.

To find the fluxion of a fraction $\frac{x}{y}$.

(17.) Put $z=\frac{x}{y}$, then $zy=x$, and $z\dot{y}+y\dot{z}=\dot{x}$ (Art. 14.); $\therefore\ \dot{z}=\frac{\dot{x}-z\dot{y}}{y}=\frac{\dot{x}-\frac{x}{y}\times\dot{y}}{y}=\frac{y\dot{x}-x\dot{y}}{y^2}$. Hence, we find the fluxion of a fraction by the following

RULE.

From the fluxion of the numerator multiplied into the denominator, subtract the fluxion of the denominator multiplied into the numerator, and divide by the square of the denominator.

EXAMPLES.

Ex. 1. The fluxion of $\frac{x^2}{y^3}$ is $\frac{2y^3x\dot{x}-3x^2y^2\dot{y}}{y^6}=\frac{2yx\dot{x}-3x^2\dot{y}}{y^4}$.

Ex. 2. The flux. of $\frac{x+y}{z^3}$ is $\frac{z^3\times(\dot{x}+\dot{y})-(x+y)\times3z^2\dot{z}}{z^6}=\frac{z\times(\dot{x}+\dot{y})-(x+y)\times3\dot{z}}{z^4}$.

Ex. 3. The flux. of $\frac{xy}{z^2}$ is $\frac{z^2\times(x\dot{y}+y\dot{x})-xy\times2z\dot{z}}{z^4}=\frac{z\times(x\dot{y}+y\dot{x})-2xy\dot{z}}{z^3}$.

Ex. 4. The fluxion of $\frac{a}{x}$ is $\frac{-a\dot{x}}{x^2}$; for a being constant, the fluxion of the numerator is nothing, and therefore the fluxion of the numerator multiplied into the denominator is nothing; in this case therefore, the fluxion of the fraction is *minus* the fluxion of the denominator multiplied into the numerator, divided by the square of the denominator.

Ex. 5. The fluxion of $\frac{1}{x^n}$ is $\frac{-nx^{n-1}\dot{x}}{x^{2n}} = -\frac{n\dot{x}}{x^{n+1}} = -nx^{-n-1}\dot{x}$; or the fluxion of $x^{-n} = -nx^{-n-1}\dot{x}$; when therefore the index of a quantity is negative, the fluxion is found by the same rule (Art. 13.) as when the index is positive.

Ex. 6. The fluxion of $\frac{\sqrt{a^2+x^2}}{\sqrt{b^2+y^2}}$ is

$$\frac{\left.\overline{a^2+x^2}\right|^{-\frac{1}{2}} \times x\dot{x} \times \sqrt{b^2+y^2} - \left.\overline{b^2+y^2}\right|^{-\frac{1}{2}} \times y\dot{y} \times \sqrt{a^2+x^2}}{b^2+y^2}$$

$$= \frac{x\dot{x}}{\sqrt{a^2+x^2} \times \sqrt{b^2+y^2}} - \frac{\sqrt{a^2+x^2} \times y\dot{y}}{\left.\overline{b^2+y^2}\right|^{\frac{3}{2}}}.$$

The putting of a quantity into fluxions, is called the *direct* method of fluxions.

SCHOLIUM.

(18.) In questions of a *geometrical* and *philosophical* nature, where we want to get the relation of the fluents from the fluxions, and in others where we want to find whether quantities are positive or negative from the relation of them to their fluxions, it is necessary to pay regard to the *signs* of the fluxions, as explained in Art. 16. But in putting equations into fluxions, as in the Problems de Maximis et Minimis, although one variable quantity may increase at the same time that another decreases, yet we may write the fluxion of each positive; for by writing it so in each equation, in order to obtain the same fluxion from the different equations, the result will not be altered. In these, and such like cases, we may therefore make the fluxion of each quantity positive, and the result will be the same. We may further observe, that when any fluxion becomes negative according to the above rule, the quantity which expresses it's value becomes negative. For instance, if $r=$ the radius of a circle, $x=$ the versed sine, $y=$ the right sine of an arc, then $y^2=2rx-x^2$, and $\dot{y}=\frac{r\dot{x}-x\dot{x}}{y}$; now for the first quadrant, x and y increase, and each fluxion is positive, and the value of $\dot{y}$ is positive, x being less than r; but in the second quadrant, y decreases and it's fluxion becomes negative, and it's value becomes negative, x being greater than r. This circumstance is similar to the case of a quantity passing through o and changing it's sign, for $\dot{y}=o$ at the end of the quadrant.

(19.) When we compare the fluxions of two quantities, by comparing the increments that would be *uniformly* generated in a given time, the quantities have been supposed to be homogeneous, there being no relation between those which are not homogeneous; yet if, of two heterogeneous quantities, the *numerical value* of one be expressed in terms of the other, it is manifest that there will be no impropriety in expressing the fluxion of one in terms of the fluxion of the other. If one side of a right-angled parallelogram be represented by 6 and the other by 9, we say, $6\times9=54$ the area; our numerical operation is perfectly correct, but no one ever imagined that the units represented by 54 are homogeneous to the units represented by 6 and 9; if 6 and 9 represent inches in *length*, 54 will represent so many *square* inches, or so many *square* areas, the side of each of which is 1 inch in length. Or if a and x represent the two sides, the area of the parallelogram will actually be ax, referring that quantity to it's proper units; although therefore there is no relation between the area and either of it's sides, yet it is expressed in terms of the sides. And if a be constant and x variable, the fluxion of the area will be $a\dot{x}$ by Prop. 3; if therefore ($\dot{x}$) the fluxion of the abscissa x be 1 inch in *length*, the corresponding fluxion of the area will be a square inches; if $\dot{x}$ be 2 inches in *length*, the fluxion of the area will be $2a$ square inches. And in general, when we consider, any two quantities which are not homogeneous, although their fluxions, which are expressed by their increments *uniformly* generated in a given time, can have no relation to each other, if we carry our ideas no further than the increments themselves; yet when we consider the numerical values of these fluxions, the analytical expression for one may be comprised in terms of the other without any impropriety, and our conclusions will be perfectly just and correct, in the sense in which the units of the respective quantities are understood, notwithstanding the fluxions themselves may be heterogeneous. Sir I. Newton, in his *Quadrature of Curves*, in finding the area of a curve, describes a parallelogram on the abscissa (x), the other side (a) of which is constant; and then he compares the fluxion of the area of this parallelogram with the fluxion of the area of the curve, they being homogeneous quantities; and the fluxion of the area of the parallelogram being $a\dot{x}$, he gets the fluxion of the area of the curve. From what has been said above, when we reduce these matters to calculation, there appears to be no absolute necessity for this; but it is more scientific to make the comparison between homogeneous quantities, than between those which are not homogeneous, and therefore the former method is always to be preferred in cases where it can be applied, notwithstanding the conclusions which are otherwise deduced are perfectly true and satisfactory.

(20.) The ingenious and justly celebrated Author of the *Analyst* has endeavoured to show, that the principles of Fluxions, as delivered by it's Author, are not founded upon reasoning strictly logical and conclusive. He lays this down as a Lemma: "If you make any supposition, and in virtue thereof deduce any consequence; if you destroy that supposition, every consequence before deduced must be destroyed and rejected, so as from thence forward to be no more supplied or applied in the demonstration." This, he thinks, is so plain as to need no proof. It may perhaps be admitted to be true, when we want to deduce the *absolute* value of a quantity which is to be obtained in virtue of a supposition; but it is not true when we want to obtain the *relative* values of quantities. He seems not to have properly attended to the meaning of the term *limiting* ratio, but went upon the term *ultimate* ratio, assuming equality where it was never intended, thereby totally misunderstanding the subject; and this led him to disregard the connection which there must necessarily be between the two terms x, y, which constitute a ratio, and the two terms m, n, which express the ratio to which x, y, approach as their limit, when you diminish them *sine limite*, called the *limit* of the ratio; for every one must see, that if you make x and y vanish, they must approach to some ratio as their limit; but we do not say (as writers who do not understand the subject would make us say) when x and y become $=0$, that $o : o :: m : n$; such is the assertion of those only who are ignorant of the subject. Now it is agreed, that by diminishing the increments you approach to the ratio of the velocities which the quantities had at the points from whence the increments began to be generated, and that by making them become indefinitely small, you arrive at a ratio indefinitely near to that of the velocities at those points. Let therefore x and y be two increments generated by two flowing quantities in the same time; then as their limit $m : n$ must depend altogether upon x and y, that *limit* is obtained upon the supposition of the existence of the increments; but the limit is a certain determinate invariable ratio, totally independent of the *magnitude* of the terms of the ratio, or of the increments, as appears by Art. 8. A ratio may *limit* the variable ratio of two increments, although it cannot be said to be the ratio of any of the real increments. When we therefore deduce the *limit* by making the increments vanish, the *effect* of the prior existence of the terms x, y, of the ratio still remains in the terms m, n, which express the *limit* of the ratio. If the *existence* of the terms m, n, which express the *limit* of the ratio, depended upon the *existence* of the terms themselves x, y, of the ratio, the supposition which makes the latter vanish would necessarily make the former also vanish, and then no conclusion could be deduced by making the terms of the ratio vanish; but as that is not the case, the *limit*, which is obtained by making the terms become equal to nothing, contains an effect, after the increments are actually vanished, which depends upon their having existed. The *limiting* ratio is (as expressed by *Maclaurin*) "the term or limit from which the variable ratio of the increments proceeds, or sets out, to increase or decrease." The lemma therefore of the Author, however true it may be under some circumstances, cannot be applied against the reasoning upon which the principles of Fluxions are founded. The Author admits the conclusions to be true. He says, "I have no controversy about your conclusions, but only about your logic; and it must be remembered, that I am not concerned about the truth of your theorems, but only about the way of coming at them." The above observations show, not only that our conclusions are true, but that they are deduced by steps which are perfectly satisfactory, and strictly logical. It was unfortunate for Science, that neither the ingenious author of the *Analyst*, nor his opponents, had any clear ideas of the subject they disputed upon; the controversy however called forth *Robins* and *Maclaurin*, who showed in the most satisfactory manner, that the grounds of fluxions, according to the ideas of it's great Author, were defensible, and the investigations founded upon the strictest principles of reasoning.

SECT. II.

ON THE MAXIMA AND MINIMA OF QUANTITIES.

PROP. IX.

To determine the value of a quantity, when it becomes a maximum or minimum.

(21.) IF a quantity first increase and then decrease, at the end of it's increase it becomes a maximum; and if it first decrease and then increase, at the end of it's decrease it becomes a minimum. And as the fluxion of a quantity is the rate of it's increase or decrease (Art. 3.), when it becomes a maximum or minimum it's fluxion must be $= 0$, the quantity having, at that point of time, no further increase or decrease. A maximum or minimum therefore do not necessarily mean the greatest or least value of a variable quantity, since, besides these values, the quantity may sometimes increase or decrease, *sine limite.* If the value which is to give the maxima or minima come out impossible, the given quantity has no maxima or minima but what are infinite.

(22.) If any quantity be a maximum or minimum, any power or root of that quantity must then, evidently, be a maximum or minimum. For the power or root of a quantity will increase or decrease as long as the quantity itself increases or decreases, and no longer.

Any constant multiple, or part of a quantity which is a maximum or minimum, must also be a maximum or minimum. For the multiple, or part of a quantity, will increase or decrease as long as the quantity itself increases or decreases, and no longer; therefore when it's fluxion is made = 0, the constant multiplier may be neglected.

EXAMPLES.

Ex. 1. *To divide a given number* a *into two parts,* x, y, *so that* $x^m y^n$ *may be a maximum.*

Since $x+y=a$, and $x^m y^n = \text{max.}$ the fluxion of each $=0$, the former, because it is constant, and the latter, because it is a maximum; $\therefore \dot{x}+\dot{y}=0$, and $my^n x^{m-1}\dot{x} + nx^m y^{n-1}\dot{y}=0$; hence, $\dot{x}=-\dot{y}$, and $\dot{x}=-\frac{nx^m y^{n-1}\dot{y}}{my^n x^{m-1}} = -\frac{nx\dot{y}}{my}$; therefore $-\dot{y}=-\frac{nx\dot{y}}{my}$; or $my=nx$, and $m:n::x:y$. Now $y=\frac{nx}{m}$; $\therefore x+\frac{nx}{m}=a$, consequently $x=\frac{ma}{m+n}$, and $y\left(=\frac{nx}{m}\right)=\frac{na}{m+n}$.

If $m=n$, the two parts are equal.

Cor. Hence, to divide a quantity a into three parts, x, y, z, so that xyz may be a max. the parts must be equal. For suppose x to have it's proper value and to remain constant, and y, z, to vary; the product yz, and consequently xyz, will be greatest when $y=z$. Or if y remain constant, the product xz, and consequently yxz, will be greatest when $x=z$. Thus it appears that the parts must be equal. And in like manner it may be shown, that whatever be the number of parts, they will be equal.

Ex. 2. *Given* $x+y+z=a$, *and* xy^2z^3 *a maximum, to find* x, y, z.

As x, y, z, must have some certain determinate values to answer these conditions, let us suppose such a value of y to remain constant, whilst x and z vary till they answer the conditions, and then $\dot{x}+\dot{z}=0$ and $z^3\dot{x}+3xz^2\dot{z}=0$; hence, $\dot{x}=-\dot{z}=-\frac{3xz^2\dot{z}}{z^3}=-\frac{3x\dot{z}}{z}$,

$\therefore z=3x$. Now let us suppose the value of z to remain constant, and x and y to vary, so as to satisfy the conditions; then $\dot{x}+\dot{y}=0$, $y^2\dot{x}+2xy\dot{y}=0$; hence, $\dot{x}=-\dot{y}=-\frac{2xy\dot{y}}{y^2}=-\frac{2x\dot{y}}{y}$, $\therefore y=2x$; substitute in the given equation, these values of y and z in terms of x, and $x+2x+3x=a$, or $6x=a$; hence, $x=\frac{1}{6}a$; $\therefore y=\frac{1}{3}a$; $z=\frac{1}{2}a$. In like manner, whatever be the number of unknown quantities, make any one of them variable with each of the rest, and the values of each in terms of that one quantity will be obtained; and by substituting the values of each in terms of that one, in the given equation, you will get the value of that quantity, and thence the values of the others.

In like manner, if $a^x b^z=A$, $(x+1)\times(z+1)=\text{max.}$ then $a^{x+1}=b^{z+1}$. For (Prop. 52.) the fluxion of $a^x b^z=a^x b^z\times(\dot{x}\log.a-\dot{z}\log.b)=0$, and $\dot{z}=\frac{\log.a}{\log.b}\dot{x}$; and the fluxion of $(x+1)\times(z+1)=x\dot{z}-z\dot{x}-\dot{x}+\dot{z}=0$; make the two values of $\dot{z}$ equal, and $(x+1)\times\log.a=(z+1)\times\log.b$, and (Art. 109.) $a^{x+1}=b^{z+1}$.

In taking the fluxions we have here observed the directions in Art. 16.

Ex. 3. *To find when* y *is a max. in* $\overline{x^3+y^3}\rvert^2=a^4x^2$.

Take the fluxions of both sides, and $2\times(3x^2\dot{x}+3y^2\dot{y})\times(x^3+y^3)=2a^4x\dot{x}$; but when y is a maximum, $\dot{y}=0$; hence, $6x^2\dot{x}\times(x^3+y^3)=2a^4x\dot{x}$, $\therefore (x^3+y^3)=\frac{a^4}{3x}$, and $(x^3+y^3)^2=\frac{a^8}{9x^2}$; therefore $a^4x^2=\frac{a^8}{9x^2}$, and $x^4=\frac{a^4}{9}$, or $x=\frac{a}{\sqrt{3}}$; hence, $y^3(a^2x-x^3)=\frac{a^3}{\sqrt{3}}-\frac{a^3}{3^{\frac{3}{2}}}=a^3\times\left(\frac{1}{\sqrt{3}}-\frac{1}{3^{\frac{3}{2}}}\right)=a^3\times\frac{2}{3\sqrt{3}}$; $\therefore y=a\sqrt[3]{\frac{2}{3\sqrt{3}}}$.

Otherwise. As $y^3=a^2x-x^3$, $\therefore\ 3y^2\dot{y}=a^2\dot{x}-3x^2\dot{x}=0$, because $\dot{y}=0$, $\therefore\ x=\frac{a}{\sqrt{3}}$.

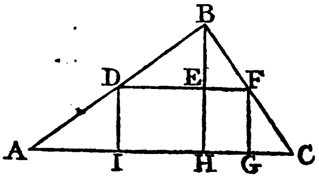
Ex. 4. *Given* (A) $ay^3-x^2y^2+x^4$ *a minimum, to find* x *and* y.

It's fluxion $(\dot{A})$ $3ay^2\dot{y}-2y^2x\dot{x}-2x^2y\dot{y}+4x^3\dot{x}=0$, and making the homologous terms (or those where $\dot{x}$, $\dot{y}$, enter) each $=0$, we have $3ay^2\dot{y}-2x^2y\dot{y}=0$, $-2y^2x\dot{x}+4x^3\dot{x}=0$, or $3ay^2=2x^2y$, and $y^2=2x^2$; hence, $x^2=\frac{3ay}{2}$ and $=\frac{y^2}{2}$, therefore $y=3a$, and $x^2=\frac{1}{2}y^2=\frac{9}{2}a^2$, or $x=\frac{3a}{\sqrt{2}}$. Substitute these for x and y into (A), and its least value comes out $=6\frac{3}{4}a^4$. We must here attend to what is said in Art. 21. respecting the minimum of a quantity.

But it is not necessary to make the homologous terms each $=0$, we may assume $3ay^2\dot{y}-2y^2x\dot{x}=0$, $4x^3\dot{x}-2x^2y\dot{y}=0$, and the condition of $(\dot{A})$ is still answered. For, $\dot{x}=\frac{3ay^2\dot{y}}{2y^2x}$ and $=\frac{2x^2y\dot{y}}{4x^3}$; and $y=3a$; substitute this for y in (A) and $27a^4-9a^2x^2+x^4=$ a min. $\therefore\ -18a^2x\dot{x}+4x^3\dot{x}=0$, and $x^2=\frac{9}{2}a^2$ as before.

Ex. 5. *To inscribe the greatest parallelogram* DFGI *in a given triangle* ABC.

Draw *BH* perpendicular to *AC*; put $AC=a$, $BH=b$,

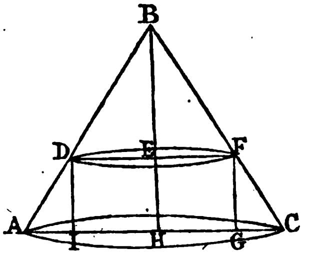


$BE=x$, then $EH=b-x$; and by sim. △s, $b:a::x:$

$DF=\frac{ax}{b}$; hence, the area $DFGI=\frac{ax}{b}\times(b-x)=$ max. or (Art. 22.) $x\times(b-x)=bx-x^2=$ max. $\therefore b\dot{x}-2x\dot{x}=0$; hence, $x=\frac{1}{2}b$; therefore $EH=\frac{1}{2}BH$.

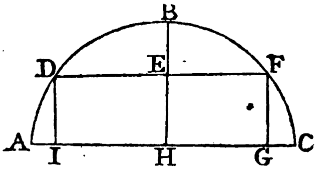
Ex. 6. *Let* ABC *represent a cone,* AC *the diameter of the base; to inscribe in it the greatest cylinder* DFGI.

Put $p=,78539$ &c. then (the same notation remaining) it will appear when we come to treat on the method of finding the areas of curves, that $\frac{pa^2x^2}{b^2}=$ the area of the end DEF of the cylinder; hence, the content of the cylinder $=\frac{pa^2x^2}{b^2}\times(b-x)=$ max. or $x^2\times(b-x)=bx^2-x^3=$ max. $\therefore 2bx\dot{x}-3x^2\dot{x}=0$; hence, $x=\frac{2}{3}b$; therefore $EH=\frac{1}{3}BH$.



Ex. 7. *To inscribe the greatest parallelogram* DFGI *in a given parabola* ABC.

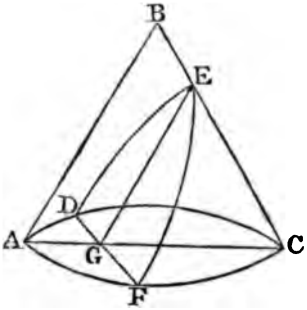
Put $BH=a$, $p=$ the parameter, $x=BE$; then by the property of the parabola, $DE^2=px$, $\therefore DE=p^{\frac{1}{2}}x^{\frac{1}{2}}$, and $DF=2p^{\frac{1}{2}}x^{\frac{1}{2}}$; hence, the area $DFGI=2p^{\frac{1}{2}}x^{\frac{1}{2}}\times(a-x)$



=max. or $x^{\frac{1}{2}} \times (a-x) = ax^{\frac{1}{2}} - x^{\frac{3}{2}}$ =max. ∴ $\frac{1}{2} ax^{-\frac{1}{2}} \dot{x} - \frac{3}{2} x^{\frac{1}{2}} \dot{x} = 0$; hence, $\frac{a}{x^{\frac{1}{2}}} = 3x^{\frac{1}{2}}$, or $a = 3x$, ∴ $x = \frac{1}{3} a$; consequently $EH = \frac{2}{3} BH$.

Ex. 8. *To cut the greatest parabola* DEF *from a given cone* ABC.

Let AGC be that diameter of the base which is perpendicular to DGF; now EG is parallel to AB;



put $AC=a$, $AB=b$, $CG=x$, then $AG=a-x$; and by the property of the circle, $DG=\sqrt{ax-x^2}$, ∴ $DF = 2\sqrt{ax-x^2}$; also, by sim. △s, $a : b :: x : GE = \frac{bx}{a}$; hence, we have the area of the parabola $= \frac{2}{3} \times \frac{bx}{a} \times 2\sqrt{ax-x^2}$ = max. hence, $x\sqrt{ax-x^2}$ = max. or $x^2 \times (ax-x^2) = ax^3 - x^4$ = max. ∴ $3ax^2\dot{x} - 4x^3\dot{x} = 0$, and $3a = 4x$, ∴ $x = \frac{3}{4} a$.

Ex. 9. *To divide a given arc* A *into two parts, such that the* m^th^ *power of the sine of one part, multiplied into the* n^th^ *power of the sine of the other, may be a maximum.*

Let P and Q be the two parts, x and y their sines, radius being unity; then $x^m \times y^n$ = maximum; hence,

$my^nx^{m-1}\dot{x}+nx^my^{n-1}\dot{y}=0$, and $my\dot{x}=-nx\dot{y}$. Now (Art. 46.) $\dot{P}=\frac{\dot{x}}{\sqrt{1-x^2}}$, $\dot{Q}=\frac{\dot{y}}{\sqrt{1-y^2}}$; and as $P+Q=A$, $\dot{P}+\dot{Q}=0$, $\therefore\ \dot{P}=-\dot{Q}$, or $\frac{\dot{y}}{\sqrt{1-y^2}}=\frac{-\dot{x}}{\sqrt{1-x^2}}$; multiply this equation by the equation $my\dot{x}=-nx\dot{y}$, and $m\times\frac{y}{\sqrt{1-y^2}}=n\times\frac{x}{\sqrt{1-x^2}}$, or $m\times$ tan. $Q=n\times$ tan. P, $\therefore m:n::$ tan. P : tan. Q, and $m+n:m-n::$ tan. $P+$ tan. Q : tan. $P-$ tan Q :: (Trig. Art. 113.) sin. $(P+Q)$: sin. $(P-Q)$ :: sin. A : sin. $(P-Q)=\frac{m-n}{m+n}\times$ sin. A; hence we know the sine of the difference of the two parts of the arc, therefore we know the difference $P-Q$ of the arcs themselves; and knowing the sum $P+Q$, or A, we know the two parts P and Q.

If the given arc be divided into three parts P, Q, R, whose sines are x, y, z, to find when $x^my^nz^r$ is a max. then by proceeding as in Ex. 2. we get the tangents as m, n, r. And the same into whatever number of parts the arc is divided.

Ex. 10. *To determine at what angle the wind must strike against the sails of a mill, so that the effect to put it in motion may be the greatest possible.*

Put $x=$ the cosine of the angle, then $1-x^2=$ the square of the sine, radius being unity; hence (by the Principles of Hydrostatics), the effect is as $x\times(1-x^2)=x-x^3$, which is to be maximum; $\therefore\ \dot{x}-3x^2\dot{x}=0$; hence, $x=\sqrt{\frac{1}{3}}$ the cosine of 54°. 44′.

Ex. 11. *Given two elastic bodies* A *and* C, *to find an intermediate body* x, *so that the motion communicated from* A *to* C *through* x, *may be a maximum.*

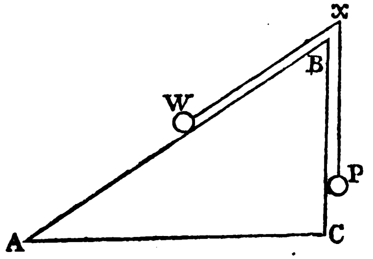
Put $a=$ the given velocity of A, $w=$ the velocity communicated to x, and z the velocity communicated to C; then (by Mechanics),

$$\begin{array}{l} A+x : 2A :: a : w \\ x+C : 2x :: w : z \\ \hline \end{array}$$

∴ comp. $Ax+x^2+AC+Cx : 4Ax :: a : z$, or $A+x+\frac{AC}{x}+C : 4A :: a : z$; now as the two middle terms are constant, the last term varies inversely as the first; and as the last is to be a maximum, the first must be a minimum; therefore its fluxion $\dot{x}-\frac{AC\dot{x}}{x^2}=0$; hence, $x^2=AC$, and $A : x :: x : C$.

√ **Ex. 12.** *Given the altitude* BC *of an inclined plane* AB, *to find it's length, so that a weight* P *acting upon another* W *in a line parallel to the plane, may draw it up through* AB *in the least time.*

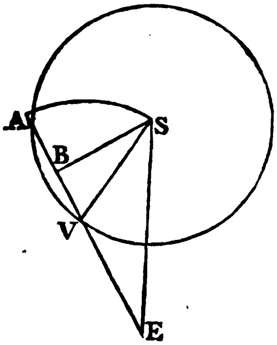
Put $a=BC$, $x=AB$; then (by Mechanics) the accelerating force of W down BA is $\frac{aW}{x}$; hence, the moving force of the two bodies is $P-\frac{aW}{x}=\frac{Px-aW}{x}$; therefore the accelerating force $=\frac{Px-aW}{(P+W)\times x}$; and the time of describing AB varies as $\sqrt{\frac{AB}{\text{ac. for.}}}$, or as $\sqrt{\frac{(P+W)\times x^2}{Px-aW}}=$ min. or $\frac{x^2}{Px-aW}=$ min. ∴



$\frac{2x\dot{x}\times(Px-aW)-P\dot{x}\times x^2}{\overline{Px-aW}|^2}=0$; but when a fraction vanishes, it's numerator $=0$; hence, $2Px^2\dot{x}-2aWx\dot{x}-Px^2\dot{x}=0$, or $Px^2=2aWx$, $\therefore x=\frac{2aW}{P}$.

Ex. 13. *To find the position of the planet Venus, when it gives the greatest quantity of light to the Earth, the orbits being supposed to be circles with the Sun in their common centre.*

Let S be the Sun, E the Earth, V Venus, produce EV, on which let fall the perpendicular SB, and with the center V describe the circular arc SA. Put $a=SE$,



$b=SV=AV$, $x=EV$, $y=BV$, then $AB=b-y$ the versed sine of the angle SVA; and (by the Principles of *Astronomy*) the quantity of light received at the Earth from Venus varies as $\frac{b-y}{x^2}=\frac{b}{x^2}-\frac{y}{x^2}=$ max. Now (Euc. B. II. p. 12.) $a^2=b^2+x^2+2xy$, $\therefore y=\frac{a^2-b^2-x^2}{2x}=$ (if $m^2=a^2-b^2$) $\frac{m^2-x^2}{2x}$; hence, the quantity of light varies as $\frac{b}{x^2}-\frac{m^2-x^2}{2x^3}=\frac{2bx-m^2+x^2}{2x^3}$, which is therefore a maximum; hence, it's fluxion $\frac{(2b\dot{x}+2x\dot{x})\times 2x^3-6x^2\dot{x}\times(2bx-m^2+x^2)}{4x^6}=0$, or it's numerator $4bx^3\dot{x}+4x^4\dot{x}-12bx^3\dot{x}+6m^2x^2\dot{x}+6x^4\dot{x}=0$, or by dividing by $2x^2\dot{x}$, and uniting the like terms, we have $-x^2-4bx+3m^2=0$, $\therefore x^2+4bx=3m^2=3a^2-3b^2$, a quadratic, from which $x=-2b+\sqrt{b^2+3a^2}$. Hence we know the three sides of the triangle ESV, to find the angle E of elongation. Now if $a=1$, $b=0,72333$ according to Dr. HALLEY; hence, $x=0,43046$, and the angle $SEV=39°.\ 44'$ the elongation of Venus from the Sun when she is brightest. Also, the angle $ESV=22°.\ 21'$; but the angle $ESV=43°.\ 40'$ at the planet's greatest elongation; hence, Venus is brightest between her inferior conjunction and her greatest elongation.

For the planet *Mercury*, $b=0,3171$, and $x=1,00058$, and the angle $SEV=22°.\ 19'$ the elongation of Mercury when brightest. Also, the angle $ESV=78°.\ 56'$; but the angle $ESV=67°.\ 13',5$ at the time of the planet's greatest elongation; hence, Mercury is brightest between it's greatest elongation and superior conjunction.

In questions of a *geometrical* and *philosophical* nature, there are frequently restrictions which do not enter into the analytical expression. In the analytical expression, considered simply as such, the unknown quantity may be assumed of any value, and therefore it may be taken *without* the limits to which it is confined by the question. When it's fluxion is therefore made equal to nothing, that equation may contain, besides the roots which are applicable to the question, others which are not applicable; and if none of the roots be applicable, it shows that the maximum or minimum of the expression do not lie within the limit of the unknown quantity, as confined by the question; in which case, the roots deduced from making the fluxion of the equation $=0$, can be of no use. In the present instance, the expression is $\frac{2bx-m^2+x^2}{2x^3}$ (A) for the quantity of light; and putting it's fluxion $=0$, we get $x=-2b\pm$

$\sqrt{b^2+3a^2}$; but it is only the root $x = -2b + \sqrt{b^2+3a^2}$ which is applicable to the question, as this is a value of x which lies within the limits of the question; and it gives the expression (A) a maximum. The other root $x = -2b - \sqrt{b^2+3a^2}$ being negative, which x never can be, cannot be applicable to the question; but it nevertheless gives the value of (A) when a minimum. But although, when we make $(\dot{A}) = 0$, the roots of the equation do not give the points in the orbit where the light is a minimum, that is, the superior and inferior conjunctions; yet if we suppose x to be confined to the limits of the question, or to represent EV, and V to move round in the circumference of the circle, in the two conjunctions $\dot{x} = 0$, and we still have $(\dot{A}) = 0$ for those points. The equation therefore $(\dot{A}) = 0$ is, under the above restrictions, true for those points, because $\dot{x} = 0$, and not because the roots give those points. Whilst, in general, a maximum or minimum of (A) lie within the value of x as restrained by the question, the roots of $(\dot{A}) = 0$ will give those points; otherwise, not; and the maximum or minimum in the question must in the latter case be sought for, by considering, when the quantity which is to be a maximum or minimum, ceases to increase or decrease, according to the restrictions of the unknown quantity. In the present instance, it is when $\dot{x} = 0$, or in the two conjunctions; for had (A) decreased and then increased between the maximum of light and either conjunction, there would have been a root of $(\dot{A}) = 0$ which would have shown the point where the light was a minimum; but as there is no such root, it shows that (A) must decrease till the planet comes into each conjunction; and as (A) then increases again by the same steps by which it decreased, the light at those points must have been a minimum. These observations appear to be of some importance, as they tend to remove difficulties which might otherwise arise in the maxima and minima of quantities which are under certain restrictions; for it might naturally be asked, in the present question for instance, why does not the equation $(\dot{A}) = 0$ give three roots, one producing a maximum and the other two the minima of light, there actually being such points in one synodic revolution of the planet?

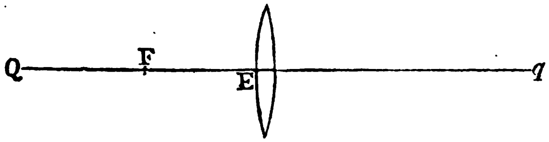
For a *superior* planet, the maximum of light is evidently when the planet is in opposition, the whole face being then illuminated, and the planet is at it's nearest distance. Now to find whether the quantity of light becomes a *minimum* in going from opposition to conjunction, we still have $x = -2b \pm \sqrt{b^2 + 3a^2}$. Now as a is less than b, $b^2 + 3a^2$ is less than $4b^2$, and $\sqrt{b^2 + 3a^2}$ is less than $2b$; hence, x $(= -2b + \sqrt{b^2 + 3a^2})$ is negative; and the other root is manifestly negative; which not being possible for x, it appears that there is no *minimum* of light in going from opposition to conjunction, but that the quantity of light continually decreases through that part of the orbit. The expression (A) does not pass through it's maximum and minimum in opposition and conjunction, for the reason before given, and therefore the roots of $(\dot{A}) = 0$ cannot give those points.

If $b = a$, $x = 0$, and V coincides with E.

It sometimes happens, that both the maximum and minimum take place when $\dot{x} = 0$ from the nature of the figure, and not from a root of the equation. Let a body move in straight lines from one focus of an ellipse to the curve and thence to the other focus, to find when the whole time will be a maximum and when a minimum, the velocity in the first line being to the velocity in the second as $a : b$. Let m = the major axis, x = the first line, $m - x$ = the second; then $a : x :: 1'' : \frac{x}{a}$ the time of describing x, and $b : m-x :: 1'' : \frac{m-x}{b}$ the time of describing $m-x$, the velocity of the body being measured by the space described uniformly in $1''$; hence, $\frac{x}{a}+\frac{m-x}{b}=$ max. or min. or $bx+ma-ax=$ max. or min. $\therefore b\dot{x}-a\dot{x}=0$, or $b\dot{x}=a\dot{x}$; but as b is not equal to a, this equation can happen only when $\dot{x}=0$, which takes place at the extremities of the major axis. If a be greater than b, the max. takes place at the further extremity from which the body set out, and the min. at the nearest extremity; the contrary if b be greater than a.

Ex. 14. *Let Q be an object placed beyond the principal focus* F *of a convex lens; to find it's position, when it's distance* Qq *from it's image* q, *is the least possible.*

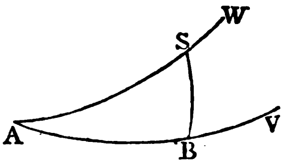
Put $QF=x$, $FE=a$; then (by the Principles of Optics) $x : x+a :: x+a : Qq=\frac{(x+a)^2}{x}=$ a min. hence,



it's fluxion $\frac{2\dot{x}\times(x+a)\times x-\dot{x}\times(x+a)^2}{x^2}=0$, and by assuming the numerator$=0$, and dividing by $x+a$, we have $2x\dot{x}-x\dot{x}-a\dot{x}=0$, or $x-a=0$, $\therefore x=a$.

Ex. 15. *To find the Sun's place in the ecliptic, when that part of the equation of time which arises from the obliquity of the ecliptic, is a maximum.*

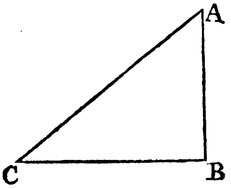
Let AV be the equator, AW the ecliptic, S the Sun's place, and SB perpendicular to AV; then this part of the equation of time is the difference of the Sun's longitude AS and right ascension AB, turned into time.



Put s=cos. of the angle $A=23°.\ 28'$, x = the tangent of AS; then by Spher. Trig. rad. $=1 : s :: x :$ tan. of $AB=sx$; hence, by Plane Trig. the tangent of $(AS-AB)$ $=\frac{x-sx}{1+sx^2}=(1-s)\times\frac{x}{1+sx^2}$ = max. or $\frac{x}{1+sx^2}$ = max. ∴ it's fluxion $\frac{\dot{x}\times(1+sx^2)-2sx\dot{x}\times x}{(1+sx^2)^2}=0$; hence, the numerator $\dot{x}+sx^2\dot{x}-2sx^2\dot{x}=0$, ∴ $1-sx^2=0$, and $x=\sqrt{\frac{1}{s}}=1{,}04416$, the tan. of 46°. 14′ the Sun's long. when this part of the equation of time is a maximum.

If we retain 1^2 in the denominator for the square of radius, as the trigonometrical theorem gives it, then $1-sx^2=0$ becomes $1^2-sx^2=0$, and $sx^2=1^2=\overline{\text{rad.}}\rceil^2$; that is, tan. $AS\times$ tan. $AB=\overline{\text{rad.}}\rceil^2$; but tan. $AS\times$ cot. $AS=\overline{\text{rad.}}\rceil^2$; therefore tan. $AB=$ cot. AS; hence, $AS+AB=90°$.

✓ Ex. 16. *Given the base* CB *of an inclined plane* AC, *to find it's altitude* BA, *when the time of the descent of a body down the plane is the least possible.*



Put $a=CB$, $x=BA$, then $\sqrt{a^2+x^2}=AC$; and (by Mechanics) the time down AC varies as $\frac{\sqrt{a^2+x^2}}{\sqrt{x}}$, which is therefore a minimum, or $\frac{a^2+x^2}{x}$ is a minimum; hence, $\frac{2x\dot{x}\times x-\dot{x}\times(a^2+x^2)}{x^2}=0$, or it's numerator $2x^2\dot{x}-a^2\dot{x}-x^2\dot{x}=0$, therefore $x^2=a^2$, and $x=a$.

Ex. 17. *Given the base* CB, *to find the perpendicular* BA, *such that a body descending from* A *to* B, *and then describing* BC *with the velocity acquired, the time through* AB *and* BC *may be the least possible.*

Put $m=16\frac{1}{12}$ feet, $a=CB$, $x=BA$; then (by Mechanics) the time down $AB=\sqrt{\frac{x}{m}}$; also, with the velocity acquired at B continued uniform, the body would describe $2AB$, or $2x$, in the same time; hence, as the space described with an uniform velocity is as the time, $2x : a :: \sqrt{\frac{x}{m}} : \frac{a}{2x}\times\sqrt{\frac{x}{m}}=\frac{1}{2}a\times\sqrt{\frac{1}{mx}}$ the time of describing BC; hence, the whole time $=\sqrt{\frac{x}{m}}+\frac{1}{2}a\sqrt{\frac{1}{mx}}=x^{\frac{1}{2}}\times\sqrt{\frac{1}{m}}+\frac{1}{2}ax^{-\frac{1}{2}}\times\sqrt{\frac{1}{m}}=$ a minimum, or $x^{\frac{1}{2}}+\frac{1}{2}ax^{-\frac{1}{2}}=$ min. $\therefore \frac{1}{2}x^{-\frac{1}{2}}\dot{x}-\frac{1}{4}ax^{-\frac{3}{2}}\dot{x}=0$, or $x^{-\frac{1}{2}}=\frac{1}{2}ax^{-\frac{3}{2}}$; hence, $x=\frac{1}{2}a$.

✓ Ex. 18. *Given the base* CB *of an inclined plane* AC, *to find it's altitude* BA, *such that the horizontal velocity of a body at* C *after descending down* AC, *may be the greatest possible.*

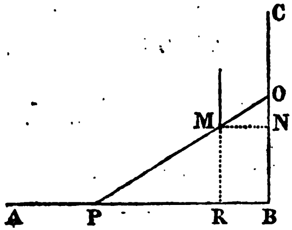
Put $a=CB$, $x=BA$, then $CA=\sqrt{a^2+x^2}$; now (by Mechanics) the velocity at C is as $\sqrt{x}$, and by the resolution of motion $\sqrt{a^2+x^2} : a :: \sqrt{x} : \frac{a\sqrt{x}}{\sqrt{a^2+x^2}}$, which is as the velocity at C in the direction BC, which is to be a maximum; or $\frac{x}{a^2+x^2}$ = a maximum; $\therefore \frac{\dot{x}\times(a^2+x^2)-2x\dot{x}\times x}{(a^2+x^2)^2}=0$, or the numerator $a^2\dot{x}+x^2\dot{x}-2x^2\dot{x}=0$; hence $x=a$.

Ex. 19. *Given the solidity of the cone, to find the base and height, when the time of it's vibration shall be a minimum, supposing the point of suspension to be the vertex.*

Put y = radius of the base, x = the altitude, p = 3,14159 &c. then $\frac{1}{3}pxy^2=s$; and (Ex. 11. Prop. 30.) $\frac{4x^2+y^2}{5x}$ = the distance from the point of suspension to the centre of oscillation = minimum. But $y^2=\frac{s}{\frac{1}{3}px}$ = (if $\frac{s}{\frac{1}{3}p}=2a$) $\frac{2a}{x}$; hence, $\frac{4x^2+\frac{2a}{x}}{5x}=\frac{4x^3+2a}{5x^2}$ = min. and $\frac{12x^2\dot{x}\times 5x^2-10x\dot{x}\times(4x^3+2a)}{25x^4}=0$; hence, $x=a^{\frac{1}{3}}$; therefore $y=\frac{\overline{2a}|^{\frac{1}{2}}}{\sqrt{x}}=\sqrt{2}\times a^{\frac{1}{3}}$; consequently $x:y::1:\sqrt{2}$.

Ex. 20. *To find the longest straight pole that can be put up a chimney.*

Let AB represent the floor, BC the back of the chimney, M the edge of the mantle-piece. Now the



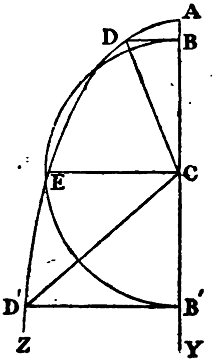
longest pole PO which can be put up, is the shortest line PMO which can be drawn through M; for any line longer than that, never can be brought into that position, but that pole can be put into any other position, every other line drawn through M being longer than that. Draw MR perpendicular to AB, and MN to BC. Put $a = MR$, $b = MN$, $x = PM$, then $PR = \sqrt{x^2 - a^2}$, and $\sqrt{x^2 - a^2} : x :: b : MO = \dfrac{bx}{\sqrt{x^2 - a^2}}$; hence, $x + \dfrac{bx}{\sqrt{x^2 - a^2}} = PO$ a min. and $\dot{x} + \dfrac{b\dot{x}\sqrt{x^2 - a^2} - bx \times \overline{x^2 - a^2}\big|^{-\frac{1}{2}} \times x\dot{x}}{x^2 - a^2} = 0$, multiply by $(x^2 - a^2) \times \sqrt{x^2 - a^2}$ and we get $(x^2 - a^2) \times \sqrt{x^2 - a^2} + bx^2 - ba^2 - bx^2 = 0$, or $\overline{x^2 - a^2}\big|^{\frac{3}{2}} = ba^2$, and $x = \sqrt{a^2 + b^{\frac{2}{3}}a^{\frac{4}{3}}}$; hence, $\dfrac{bx}{\sqrt{x^2 - a^2}} = \sqrt{b^2 + a^{\frac{2}{3}}b^{\frac{4}{3}}}$, and $PO = \sqrt{a^2 + b^{\frac{2}{3}}a^{\frac{4}{3}}} + \sqrt{b^2 + a^{\frac{2}{3}}b^4}$ the length of the pole.

If $a = b$, $PO = 2\sqrt{2a}$.

Ex. 21. *Let* A *be the vertex of a parabola* AZ *whose axis* AY *is perpendicular to the horizon,* C *any given point in* AY; *to find the line* DC *of quickest descent from the curve to* C, *and* CD′ *the quickest descent from* C *to the curve.*

Draw DB, $D'B'$, CE, perpendicular to AY, put $AC = a$, $p =$ parameter, $x = AB$, $y = BD$, then $px = y^2$, $pa = CE^2$; also, $CB = a - x$; and by Mechanics, the time down DC is as $\dfrac{DC}{\sqrt{BC}} = \dfrac{\sqrt{px + (a-x)^2}}{\sqrt{a-x}} =$ a min. or $\dfrac{px + (a-x)^2}{a-x} =$ min. or $\dfrac{px}{a-x} + a - x =$ min. $\therefore \dfrac{p\dot{x} \times (a-x) + px\dot{x}}{(a-x)^2} - \dot{x} = 0$, or $(a-x)^2 = pa$ and $a - x = \sqrt{pa}$, $\therefore x = a - \sqrt{pa}$, or $AB = AC - CE$.

In like manner, if $x = AB'$, we find $x = AC + CE$, or $AB' = AC + CE$. But $AB = AC - CB$, and $AB' = AC + CB'$; $\therefore CB' = CE = CB$. Hence, with the center C and radius CE describe a circle cutting CY in B, B', and draw the ordinates BD, $B'D'$, and DC, CD' are the lines required.



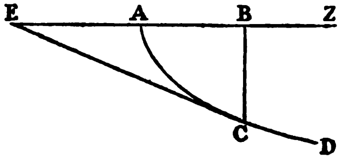
When CE becomes equal to or less than AC, AC will be the line of the shortest descent to C.

Ex. 22. *To find* x^s = *maximum, when* $x^s = \frac{1}{x^x}$.

Let m = hyp. log. x; then as x^s = max. it's hyp. log. is a max. or $ms = \frac{m}{x^x}$ = max. and the hyp. log. $\frac{m}{x^x}$ = max. that is, hyp. log. m — hyp. log. x^x, or hyp. log. $m - mx$ = max.; hence, $\frac{\dot{m}}{m} - m\dot{x} - x\dot{m} = 0$. But (Art. 45.) $\dot{m} = \frac{\dot{x}}{x}$; therefore $\frac{\dot{x}}{mx} - \dot{x} - m\dot{x} = 0$; hence, $m^2x + mx = 1$. Put $m = v - \frac{1}{2}v^2 + \frac{1}{3}v^3 -$ &c. then (Art. 103.) $(v - \frac{1}{2}v^2 + \frac{1}{3}v^3 - \text{\&c.})^2 + (v - \frac{1}{2}v^2 + \frac{1}{3}v^3 - \text{\&c.}) \times (1 + v) = 1$; and by the reversion of series (*Alg.* Art. 343.), $v = .56$, and $x = 1.56$ the required value to produce x^s a maximum.

Ex. 23. *If* AD *be a parabola whose vertex is* A, *and axis* AZ *parallel to the horizon; to find at what point* C *a body falling from* A *will strike an horizontal plane with the greatest force.*

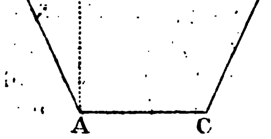
Draw the tangent EC meeting the axis produced; put $x = AB$, $y = BC$, $p =$ parameter, then $EC =$



$\sqrt{4x+y^2} = \sqrt{4x^2+px}$; and if $m = 16\frac{1}{12}$ feet, the velocity at $C = \sqrt{4my} = \sqrt{4mp^{\frac{1}{2}}x^{\frac{1}{2}}}$, and this velocity is in the direction EC, hence, $\sqrt{4x^2+px} : \sqrt{px} :: \sqrt{4mp^{\frac{1}{2}}x^{\frac{1}{2}}} : \frac{\sqrt{4mp^{\frac{3}{2}}x^{\frac{3}{2}}}}{\sqrt{4x^2+px}}$ the velocity in the direction BC, which is to be a max. or $\frac{x^{\frac{1}{2}}}{4x+p} =$ max. hence, $\frac{\frac{1}{2}x^{-\frac{1}{2}}\dot{x}\times(4x+p) - 4x^{\frac{1}{2}}\dot{x}}{(4x+p)^2} = 0$, and $x = \frac{1}{4}p$; therefore B is the focus of the parabola.

√ Ex. 24. *Given the length of a man's foot, and the distance of his heels; to find the position of his feet when he stands the firmest.*

Let AB, CD represent his feet, A, C, his heels, then



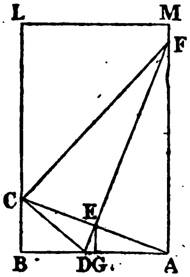
he is supposed to stand firmest when the area $ABDC$

is the greatest. Draw AE perpendicular to BD; put $AC=a$, $AB=b$, $BE=y$, then $AE=\sqrt{b^2-y^2}$, and the area $ABDC=(a+y)\times\sqrt{b^2-y^2}=\text{max.}$ hence, $\dot{y}\sqrt{b^2-y^2}-\frac{ay\dot{y}+y^2\dot{y}}{\sqrt{b^2-y^2}}=0$, or $b^2-y^2-ay-y^2=0$, and $y=-\frac{1}{4}a+\sqrt{\frac{1}{2}b^2+\frac{1}{16}a^2}$. Hence, $b:-\frac{1}{4}a+\sqrt{\frac{1}{2}b^2+\frac{1}{16}a^2}$ :: rad. : sin. BAE.

If $a=0$, or the heels touch, rad. : sin. BAE :: $1:\sqrt{\frac{1}{2}}$; hence, $BAE=45°$, and the feet stand at right angles to each other.

Ex. 25. *To turn down the corner of the leaf of a book to the back, so that the part turned down may be a minimum.*

Let $ABLM$ be the leaf, ADF the triangle turned down, so that A may fall at C in LB; join CF, DF, bisecting AC in E, and draw EG perpendicular to AB, then $AG=\frac{1}{2}AB$. Put $AG=a$, $AD=x$, then $EA=a^{\frac{1}{2}}x^{\frac{1}{2}}$, $ED=x^{\frac{1}{2}}\times\overline{x-a}|^{\frac{1}{2}}$; and by similar triangles,



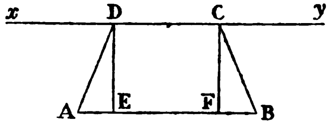
$$x^{\frac{1}{2}}\times\overline{x-a}|^{\frac{1}{2}}:x::x:DF=\frac{x^{\frac{3}{2}}}{\overline{x-a}|^{\frac{1}{2}}};$$ hence, the area

$ADF = \frac{\frac{1}{2}a^{\frac{1}{2}}x^2}{\overline{x-a}|^{\frac{1}{2}}}$ = min. or $\frac{x^4}{x-a}$ = min. therefore $\frac{3x^4\dot{x} - 4ax^3\dot{x}}{\overline{x-a}|^2} = 0$, and $x = \frac{4a}{3}$; hence, $AD : AE ::$ $2 : 1$, and the angle $CAB = 30°$.

To find when the length of the line DF is the least, we have $\frac{x^{\frac{3}{2}}}{\overline{x-a}|^{\frac{1}{2}}}$ = min. and hence, $x = \frac{3a}{2}$ and the angle $CAD =$ 35°. 16′.

√ Ex. 26. *Let* AB *be a given straight line to which* xy *is parallel; draw the straight lines* AD, BC, *and let a body move through* AD, DC, CB, *with given velocities* a, b, c, *respectively, in the least time possible; to find the points* D, C.

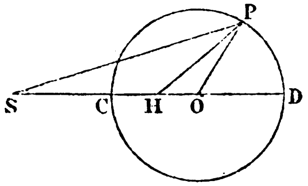
Put $x = AD$, $y = DC$, $z = CB$, and draw DE, CF, perpendicular to AB, and let $AB = m$, $DE = CF = d$;



then $AE = \sqrt{x^2 - d^2}$, $FB = \sqrt{z^2 - d^2}$, and $\sqrt{x^2 - d^2} + y + \sqrt{z^2 - d^2} = m$; also, $a : x :: 1'' : \frac{x}{a}$ the time through AD, and in like manner, $\frac{y}{b}$, $\frac{z}{c}$, are the times through DC, CB respectively; hence, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c}$ = min. make y and x vary together whilst z remain constant, and we get $\frac{\dot{x}}{a}+\frac{\dot{y}}{b}=0$, $\frac{x\dot{x}}{\sqrt{x^2-d^2}}+\dot{y}=0$, $\therefore \dot{y}=\frac{-b\dot{x}}{a}=\frac{-x\dot{x}}{\sqrt{x^2-d^2}}$, and $x=\frac{bd}{\sqrt{b^2-a^2}}$; and making y and z vary together, we get $z=\frac{bd}{\sqrt{b^2-c^2}}$. Now it is manifest that *E*, *F*, must lie between *A* and *B*; but if *a* and *c* be each $= b$, *x* and *z* become infinite; *a* and *c* must therefore be limited in respect to *b*. Now $AE\left(=\sqrt{x^2-d^2}\right)=\frac{ad}{\sqrt{b^2-a^2}}$, $DF\left(=\sqrt{z^2-d^2}\right)=\frac{bd}{\sqrt{b^2-c^2}}$; hence, $\frac{ad}{\sqrt{b^2-a^2}}+y+\frac{bd}{\sqrt{b^2-c^2}}=m$, and $\frac{ad}{\sqrt{b^2-a^2}}+\frac{bd}{\sqrt{b^2-c^2}}=m-y$. It follows therefore, that *a* and *c* must be so limited as to give $\frac{ad}{\sqrt{b^2-a^2}}+\frac{bd}{\sqrt{b^2-c^2}}$ a quantity equal to or less than *m*.

LEMMA. *Take* SC : CH :: a : b, *and* CO : SC :: CH : SC − CH; *and with the center* O *and radius* OC *describe a circle* CPD; *then* SP : HP :: a : b *wherever* P *is taken.*

From the last proportion *CO* : *CH* :: *SO* : *SC*; hence, *CO* (*PO*) : *SO* :: *HO* : *CO* (*PO*), therefore



the sides of the triangles *POH*, *POS*, about the common angle *O* being proportional, the triangles are similar, and *SP* : *HP* :: *SO* : *PO* (*CO*) :: *SC* : *CH* :: *a* : *b*.

Ex. 27. *Let there be two lights at* S *and* **H** *in the ratio of* m : 1 ; *to find the point* C *between them where the least light is received; and the locus of all the points where a proportional quantity is received.*

The quantities of light received at *C* from *S* and *H* are as $\frac{m}{SC^2} : \frac{1}{HC^2}$; hence, $\frac{m}{SC^2}+\frac{1}{HC^2}=\text{min.}$ therefore $\frac{-m\,S\dot{C}}{SC^3}-\frac{H\dot{C}}{HC^3}=0$, but as $SC+HC$ is constant, $S\dot{C}+H\dot{C}=0$, and $S\dot{C}=-H\dot{C}$; hence, $\frac{m}{SC^3}=\frac{1}{HC^3}$, and $SC : HC :: m^{\frac{1}{3}} : 1$. Having therefore determined *C*, take $CO : SC :: HC : SC-HC$, and with the center *O* describe the circle *CPD*, and it's periphery will be the locus required. For by the Lemma, $SP : HP :: SC : HC$, $\therefore \frac{1}{SC^2} : \frac{1}{HC^2} :: \frac{1}{SP^2} : \frac{1}{HP^2}$, and $\frac{m}{SP^2} : \frac{1}{HP^2} :: \frac{m}{SC^2} : \frac{1}{HC^2}$ which are as the quantities of light from *S* and *H* at *C*.

When a quantity is a maximum or minimum, it frequently shortens the operation to assume it's logarithm a max. or min. *For example,* to find when $\sqrt{x^2-ax+b}\times\sqrt[3]{m-x^3}$ is a max. or min. assume log. $\sqrt{x^2-ax+b}\times\sqrt[3]{m-x^3}$ a max. or min., or log. $\sqrt{x^2-ax+b}$ + log. $\sqrt{m-x^3}$ = max. or min.; hence, $\frac{1}{2}\times\frac{2x\dot{x}-a\dot{x}}{x^2-ax+b}-\frac{1}{3}\times\frac{3x^2\dot{x}}{m-x^3}=0$.

Ex. 28. *To find when* (A) $x^3-18x^2+96x-20$ *becomes a maximum or minimum.*

Assume the fluxion = 0, and $3x^2\dot{x}-36x\dot{x}+96\dot{x}=3\dot{x}\times(x^2-12x+32)=0$; hence, $x=4$ or 8. Now to determine which value gives the maximum and which the minimum, find whether the value of the fluxion, just before it becomes = 0, be *positive* or *negative*;

if *positive*, the succeeding root gives a *maximum*; if *negative* a *minimum*; for whilst a quantity increases it's fluxion is positive; but when it decreases it's fluxion becomes negative, by Art. 16. Now as $3\dot{x}\times(x-4)\times(x-8)=3\dot{x}\times(x^2-12x+32)$; when x is less than 4, each factor being negative, the value of the fluxion is positive, therefore the root 4 gives (*A*) $x^3-18x^2+96x-20$, a maximum; and as, when x increases from 4 to 8, one factor is positive and the other negative, the fluxion is negative, therefore the root 8 gives (*A*) a minimum. When we say that by making $x=4$ it gives (*A*) a maximum, we mean that (*A*) first increases till x becomes 4 and then it decreases, and not that it is then the greatest possible; for by increasing x after it exceeds 8, the value of (*A*) increases *sine limite*. And in like manner, (*A*) decreases whilst x increases from 4 to 8, and then it increases, and therefore when $x=8$, (*A*) is said to be a minimum, not that it is then the least possible, for by decreasing x below 4, (*A*) will decrease *sine limite*.

We have here supposed x to increase; if we suppose x to decrease, and first assume it greater than 8, then as x decreases till it becomes 8, each factor $x-4$, $x-8$ being positive, the product is positive, and therefore it might appear that the root 8 ought to give a maximum; but as x is a *decreasing* quantity, it's fluxion ($\dot{x}$) is negative by Art. 16; hence, $3\dot{x}\times(x-4)\times(x-8)$ is negative till x becomes 8, and therefore this root gives (*A*) a minimum; and whilst x decreases from 8 to 4, $3\dot{x}\times(x-4)\times(x-8)$ is positive, and therefore 4 gives (*A*) a maximum, agreeable to what was before determined. This instance shows the necessity of attending to the signs of the fluxions of increasing and decreasing quantities, without which we might have determined (*A*) to have been a maximum when it is a minimum, and a minimum when it is a maximum; for it is merely arbitrary whether we suppose x to increase or decrease.

When all the roots of the fluxional equation are impossible, as no possible value of x can make the equation $= 0$, it shows that by increasing x, the given quantity increases or decreases *sine limite*, therefore it admits of no maximum or minimum.

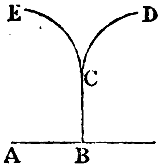
It may happen that the fluxion may be $= 0$, and yet the quantity (A) may not be a maximum or minimum, which takes place when two of the roots of the fluxional equation are *equal*, because in that case, the sign of the fluxion is the same both before and after the equation becomes $= 0$ from the substitution of one of the equal roots. For let the given quantity be $x^4 - 16x^3 + 90x^2 - 216x$, whose fluxion is $4x^3\dot{x} - 48x^2\dot{x} + 180x\dot{x} - 216\dot{x} = 4\dot{x} \times (x^3 - 12x^2 + 45x - 54) = 4\dot{x} \times (x-3) \times (x-3) \times (x-6)$. Now just before $x = 3$, this fluxion is negative, and just after $x = 3$, it is also negative; therefore as the fluxion continues negative whilst x passes through 3, that root does not give (A) a minimum; but as the fluxion passes from negative to positive whilst x passes from less than 6 to more than 6, the root 6 gives (A) a minimum, it's fluxion after that time being positive, shows that (A) then begins to increase.

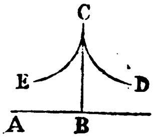
Let the fluxional equation have three equal roots, as in $\dot{x} \times (x-a) \times (x-a) \times (x-a) \times (x-b)$, and let a be less than b. Then it is manifest, that when x is less than a, this fluxion is positive, and when x passes through a and lies between a and b, the fluxion is negative; therefore $x = a$ gives (A) a maximum. Hence it is manifest, that, in general, when the fluxional equation has an *even* number of equal roots, one of those roots gives (A) neither a maximum nor minimum; but when it has an *odd* number, that root gives (A) either a maximum or minimum.

Ex. 29. *To find the value and position of the greatest and least ordinates of a curve, whose equation is* $y = x^3 - px^2 + qx - r$, x *being the abscissa and* y *the ordinate.*

Take the fluxion, and $\dot{y} = 3x^2\dot{x} - 2px\dot{x} + q\dot{x}$; but when y becomes a max. or min. $\dot{y}=0$; hence, $3x^2\dot{x}-2px\dot{x}+q\dot{x}=0$; consequently $x=\frac{p}{3}\pm\sqrt{\frac{p^2}{9}-\frac{q}{3}}$, the values of the abscissa corresponding to the required ordinates; and if these values of x be respectively substituted into the given equation, the values of the ordinates themselves will be known. Which of the values of x gives the ordinate a maximum and which a minimum, may be found by Ex. 28. If $p=18$, $q=60$, $r=10$, then $x=2$ and 10, the two abscissæ; which substituted for x in the given equation, give 46 and -210 for the two ordinates, the latter of which being negative, shows that the curve at that point lies below the abscissa. At the greatest and least ordinates the tangent is parallel to the abscissa (Art. 23).

But there are other cases when the ordinate becomes a maximum or minimum, that is, at the point C of a curve, where the ordinate BC is a tangent at C to the two parts CE, CD of the curve continued on both sides of C. When EC, CD, are *concave* to the abscissa, BC is a *minimum*, when *convex*, a maximum. In cases of this kind, we must find when the fluxion of the ordinate becomes equal to the fluxion of the curve; or, when the fluxion of the ordinate becomes indefinitely greater than the fluxion of the abscissa.





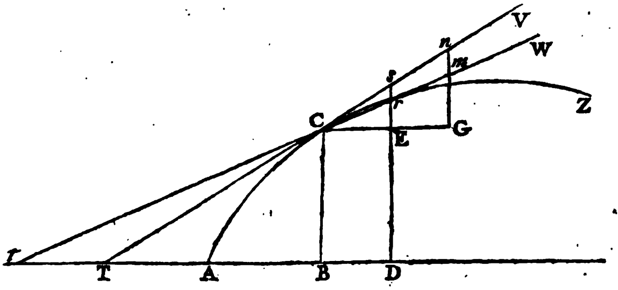
These are the rules for finding the maxima and minima when the tangents are parallel or perpendicular to the abscissa. But if C should be the point of contrary flexure of a curve, and a tangent at C should be parallel or perpendicular to the abscissa, these rules will not hold, because the ordinates will then continue to increase through the point C.

To draw TANGENTS to CURVES.

Prop. X.

Let the curve ACZ *be described by the extremity of the ordinate* BC, *which moves parallel to itself and varies in it's length; to draw a tangent to the curve at any point* C.

(23.) Let TCV be the required tangent; draw any other ordinate Dr and produce it to s; draw also CE parallel to BD; join Cr, and produce it to t and W; produce also CE to any point G, and draw Gmn parallel to Es. Now let Drs move up to BC, then by the motion of r, the line $WrCt$ will revolve about C, and when r coincides with C, it ceases to cut the curve between C and Z, and it does not cut it between C and A, for to cut CA, Ct must fall below CT, and consequently CW must lie above CV, or r must have passed s, which it cannot have done, as r has been continually approaching to s and only now coincides with it; therefore when r comes to C, the line Wt,



ceasing to cut the curve, must become a tangent, and consequently *WCt* will then coincide with *VCT*. Now whilst the abscissa *AB* by increasing becomes *AD*, the ordinate *BC* becomes *Dr*; hence, the increment of the ordinate *BC* is *Er*; and, by similar triangles, the increment *CE* of the abscissa : the cotemporary increment *Er* of the ordinate :: *CG* : *Gm*. But when *r* arrives at *C*, *WC* coincides with *VC*, and consequently *m* must coincide with *n*; hence, the *limiting* ratio of the increment *CE* of the abscissa to the increment *Er* of the ordinate, is that of the finite lines *CG* : *Gn*, which (by sim. trian.) is the ratio of *CE* : *Es*, taking *DEs* in any situation before it's coincidence with *BC*; hence, by Proposition II., if *CE* represent the fluxion of the abscissa, *Es* will represent the cotemporary fluxion of the ordinate. Put $AB = x$, $BC = y$, then $BD = CE = \dot{x}$, $Es = \dot{y}$; and as *BC* is parallel to *Es*, and *TB* to *CE*, the angle $TCB = CsE$, and $CTB = sCE$, consequently the triangles *TBC*, *CEs* are similar; hence, $\dot{y}$ (*Es*) : $\dot{x}$ (*CE*) :: y (*CB*) : $BT = \frac{y\dot{x}}{\dot{y}}$; therefore set off $BT = \frac{y\dot{x}}{\dot{y}}$, join *T* and *C*, and *TC* will be a tangent to the curve at *C*. If y decrease whilst x increases, then $\dot{y}$ becomes negative by Art. 16. and consequently $\frac{y\dot{x}}{\dot{y}}$, or *BT*, becomes negative, which shows that *T* lies on the other side of *B*. See *Algebra*, Art. 474.

When $\dot{y} = 0$, *BT* becomes infinite, and the tangent becomes parallel to the abscissa.

DEF. The line *BT* is called the *subtangent*.

EXAMPLES.

Ex. 1. *Let the curve* AC *be a* parabola, *that is, a curve whose abscissa varies as any direct power of the ordinate; to draw a tangent at the point* C.

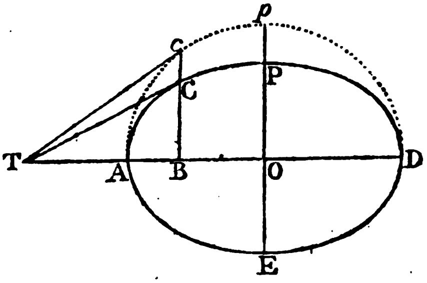
The equation expressing the relation between x and y is $ax = y^n$, for then $x : y^n :: 1 : a$, a constant ratio.

Take the fluxion of both sides of the equation, and we have $a\dot{x} = ny^{n-1}\dot{y}$; hence, $\frac{\dot{x}}{\dot{y}} = \frac{ny^{n-1}}{a}$, ∴ $BT = \frac{y\dot{x}}{\dot{y}} = \frac{ny^{n}}{a} = nx$, because $\frac{y^{n}}{a} = x$.

If $n = 2$, it is the common parabola, and $BT = 2x$.

Ex. 2. *To draw a tangent to the* ellipse ACPDE, *at any point* C.

Let AD and PE be the two axes; put $AO = a$, $PO = b$, $AB = x$, $BC = y$, then $BD = 2a - x$; and by the property of the ellipse, $a^2 : b^2 :: (2a - x) \times x : y^2 = \frac{b^2}{a^2} \times (2ax - x^2)$; take the fluxions, and $\frac{b^2}{a^2} \times (2a\dot{x} - 2x\dot{x}) = 2y\dot{y}$, multiply both sides by $\frac{a^2}{b^2}$, divide by 2 which is common, and also by $a - x$, and $\dot{x} = \frac{a^2}{b^2} \times \frac{y\dot{y}}{a - x}$, ∴ $\frac{\dot{x}}{\dot{y}} =$



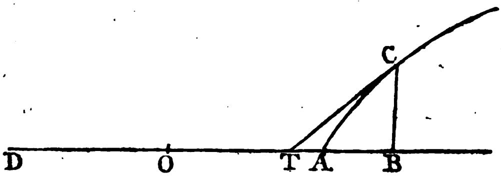
$\frac{a^2}{b^2} \times \frac{y}{a - x}$; hence, $BT = \frac{y\dot{x}}{\dot{y}} = \frac{a^2}{b^2} \times \frac{y^2}{a - x} = \frac{2ax - x^2}{a - x}$, by substituting $\frac{b^2}{a^2} \times (2ax - x^2)$ for y^2.

As this value of TB is independent of b, or PO, if we take $pO = AO$, so that ApD may be a circle, and produce BC to c, cT will be a tangent to the circle. If B be between O and D, so that whilst x increases y decreases, then $\dot{y}$ becomes negative by Art. 16. and consequently $\frac{y\dot{x}}{\dot{y}}$ is negative, which shows that the subtangent BT lies the other way from B.

If $x=BO$, then $BC=\sqrt{a^2-x^2}$, and $a^2 : b^2 :: a^2-x^2 : y^2=\frac{b^2}{a^2}\times(a^2-x^2)$; this equation is sometimes more convenient than the other.

Ex. 3. *To draw a tangent to the* hyperbola AC, *whose major axis is* AD.

Bisect AD in O; put $AO=a$, the semi-axis minor $=b$, $AB=x$, $BC=y$; then by the property of the hyperbola, $a^2 : b^2 :: (2a+x)\times x : y^2=\frac{b^2}{a^2}\times(2ax+x^2)$, which is the same equation as for the ellipse, except that the sign of x^2 is here positive; $\therefore BT=\frac{2ax+x^2}{a+x}$.



If $BO=x$, then $y^2=\frac{b^2}{a^2}\times(x^2-a^2)$.

Ex. 4. *To draw a tangent to the* Cissoid *of* Diocles, *whose equation is* $y^2=\frac{x^3}{a-x}$ (Pr. 20. Ex. 7.)

Take the fluxion, and $2y\dot{y}=\frac{3x^2\dot{x}\times(a-x)+x^3\dot{x}}{(a-x)^2}=$

$\dfrac{3ax^2\dot{x}-2x^3\dot{x}}{(a-x)^2}$; hence, $\dfrac{\dot{x}}{\dot{y}}=\dfrac{2y\times(a-x)^2}{3ax^2-2x^3}$; $\therefore BT=\dfrac{y\dot{x}}{\dot{y}}=$ $\dfrac{2y^2\times(a-x)^2}{3ax^2-2x^3}=\dfrac{2x^3}{a-x}\times\dfrac{(a-x)^2}{3ax^2-2x^3}=\dfrac{2x\times(a-x)}{3a-2x}$.

Ex. 5. *To draw a tangent to the* catenary *curve.*

The equation of this curve is $a\dot{x}=z\dot{y}$ (Prop. 130.); hence, $BT=\dfrac{y\dot{x}}{\dot{y}}=\dfrac{zy}{a}=y\dfrac{\sqrt{2ax+x^2}}{a}$.

Ex. 6. *To draw a tangent to the* logarithmic *curve.*

Here the equation is $a^x=y$ (Art. 109.); and if A and Y be the hyp. logs. of a and y; then $xA=Y$; hence, $A\dot{x}=\dot{Y}=\dfrac{\dot{y}}{y}$ (Art. 45.), therefore $BT'=\dfrac{y\dot{x}}{\dot{y}}=\dfrac{1}{A}$.

Ex. 7. *To draw a tangent to the curve whose equation is* $x^x=y$.

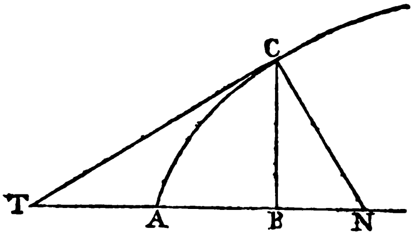
If X and Y be the hyp. logs. of x and y, we have $xX=Y$, and $x\dot{X}+X\dot{x}=\dot{Y}$; but (Art. 45.) $\dot{X}=\dfrac{\dot{x}}{x}$ and $\dot{Y}=\dfrac{\dot{y}}{y}$; $\therefore \dot{x}+X\dot{x}=\dfrac{\dot{y}}{y}$, or $y\dot{x}+yX\dot{x}$ $=\dot{y}$; hence, $BT=\dfrac{y\dot{x}}{\dot{y}}=\dfrac{y\dot{x}}{y\dot{x}+yX\dot{x}}=\dfrac{1}{1+X}$.

Ex. 8. *To draw a tangent to an* hyperbola *between the asymptotes.*

Here $xy=a^2$, therefore (Art. 16.) $y\dot{x}-x\dot{y}=0$, and $y\dot{x}=x\dot{y}$; hence $BT=\dfrac{y\dot{x}}{\dot{y}}=x$, which being negative (because when $\dot{x}$ is $+$, $\dot{y}$ is $-$), shows that T lies on the other side of the ordinate in respect to the abscissa.

(24.) Draw *CN* perpendicular to the tangent, and it is called the *normal*, and *NB* the *sub-normal.*

Now the triangles *TBC*, *NBC* are similar; hence, $\frac{y\dot{x}}{\dot{y}}$ (*TB*) : y (*BC*) :: y : $BN = \frac{y\dot{y}}{\dot{x}}$ the *sub-normal*. Also,



$$CN^2 = y^2 + \frac{y^2\dot{y}^2}{\dot{x}^2} = y^2 \times \left(1+\frac{\dot{y}^2}{\dot{x}^2}\right) = y^2 \times \frac{\dot{x}^2+\dot{y}^2}{\dot{x}^2};$$

hence, $CN = y \times \frac{\sqrt{\dot{x}^2+\dot{y}^2}}{\dot{x}}$ the *normal*.

Ex. *Let the curve be a* parabola.

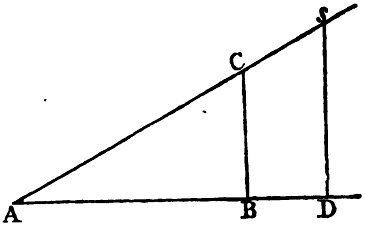
Here $ax = y^n$; ∴ $a\dot{x} = ny^{n-1}\dot{y}$, and $\frac{\dot{x}}{\dot{y}} = \frac{ny^{n-1}}{a}$, ∴ $BN = \frac{y\dot{y}}{\dot{x}} = \frac{a}{ny^{n-2}}$. In the common parabola, where $n = 2$, $BN = \frac{a}{2}$, a being the *latus rectum*. Also, $CN = \sqrt{y^2 + \frac{1}{4}a^2}$.

In an *equilateral hyperbola*, the sub-normal = *BO* (Fig. Ex. 3.).

(25.) If two quantities begin together and increase uniformly, one by x and the other by mx, m being constant, then, by the Composition of Ratios, the quantities generated will be in the ratio of $x : mx$, or as $1 : m$, a constant ratio.

(26.) If *BC* move parallel to itself, and *AB* and *BC* increase uniformly, the locus of the point

C is a straight line. For let *BC* come into the position *Ds*; then as *AB* and *BC* begin together and



increase uniformly, they have always a constant ratio to each other, by Art. 25; therefore *AB* : *BC* :: *AD* : *Ds*, which is the property of similar triangles; hence, *ACs* is a straight line. Also, as *BC* is parallel to *Ds*, *AB* : *AC* :: *BD* : *Cs*; but *AB* : *AC* in a constant ratio; if therefore *BD* the increment of the base be constant, the cotemporary increment *Cs* of the hypothenuse must be constant, or if the former increase uniformly, the latter will increase uniformly. Hence, the two *uniform* motions of *C*, one in a direction parallel to *AB* arising from the motion of *BC*, and the other in the direction *BC*, generate an *uniform* motion in a *right* line *AC*.

(27.) The fluxion of the curve line *AC*, cotemporary with *CE*, *Es* (figure to Art. 23.) the fluxions of the abscissa and ordinate, is the space that would be described by the point *C* with it's motion continued uniform for the time in which *CE*, *Es* are described. Now the motion of *C* arises from two motions, one by which it is carried parallel to *AB* by the motion of *BC*, and the other by which it is carried in the direction *BC* by the increase of *BC*; and (Art. 26.) the uniform motion of *C* is determined by making these two motions become uniform; but when these two motions become *uniform*, they are represented by *CE* and *Es*, by Art. 23. and these two *uniform* motions produce a cotemporary *uniform* motion *Cs*, by Art. 26; hence, by Prop. 1. *Cs* will represent the cotemporary fluxion of the curve line at the point *C*.

To DRAW ASYMPTOTES TO CURVES.

DEFINITION.

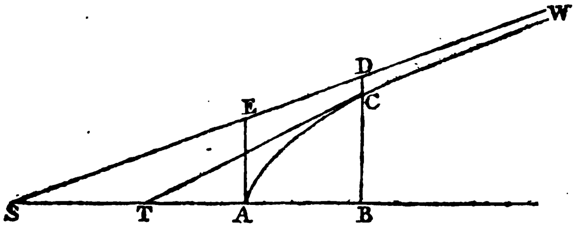
(28.) If a right line, intersecting the axis of a curve at a finite distance, continually approach the curve, and arrive nearer to it than by any assignable distance, but indefinitely produced never meets it, it is called an *Asymptote.*

Prop. XI.

To draw an asymptote to a curve.

(29.) Let *SDW* be an asymptote to the curve *AC*; then, by the definition, we may consider the asymptote *SW* as the limit to which the tangent approaches, when the abscissa *AB* is increased *sine limite.* Draw *AE* parallel to the ordinate *BC* produced to *D*, and let *TC* be a tangent to the curve at *C.*

Put $AB=x$, $BC=y$; then by Art. 23. $BT=\frac{y\dot{x}}{\dot{y}}$; hence, $AT=\frac{y\dot{x}}{\dot{y}}-x$. From the equation of the curve, find the value of this quantity when x and y are infinite, and if it then be finite, the curve admits of an asymptote *SW*, and the value of *AS* is obtained.



Then having computed the value of *BT*, find the proportion of TB to BC; and to get their limit, make x and y infinite, and you get the proportion of SB to BD, because the limit of TB to BC is SB to BD; but, by similar triangles, $SB : BD :: SA : AE$, the ratio therefore of SA to AE is known, and as AS is known, AE is known; therefore the point E is determined; draw SE, and produce it indefinitely, and it will be the asymptote.

EXAMPLES.

Ex. 1. *Let* AC *be the common* hyperbola.

Here, by Ex. 3. Art. 23. $BT = \frac{2ax+x^2}{a+x}$, therefore $AT = \frac{2ax+x^2}{a+x} - x = \frac{ax}{a+x}$, the limit of which, when x is infinite, is $\frac{ax}{x} = a = AS$; hence, S is the centre of the hyperbola. Now $BC = \frac{b}{a} \times \sqrt{2ax+x^2}$, and $BT = \frac{2ax+x^2}{a+x}$; hence, $BT : BC :: \frac{2ax+x^2}{a+x} : \frac{b}{a} \times \sqrt{2ax+x^2}$, the limit of which (when x becomes infinite) is as $x : \frac{b}{a} \times x :: a : b :: BS : BD :: AS : AE$; but $AS = a$, $\therefore AE = b$; hence, draw AE parallel to BC, and take it $= b$, join SE, and produce it indefinitely, and it will be the asymptote.

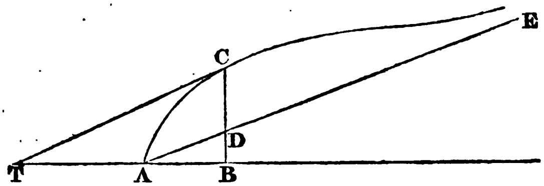
Ex. 2. *Let the equation of the curve be* $y^3 = ax^2 + x^3$.

Here $3y^2\dot{y} = 2ax\dot{x} + 3x^2\dot{x}$, and $BT = \frac{y\dot{x}}{\dot{y}} = \frac{3y^3}{2ax+3x^2} = \frac{3ax^2+3x^3}{2ax+3x^2}$; also, $BC = y = \sqrt[3]{ax^2+x^3}$; hence, $BT : BC :: \frac{3ax^2+3x^3}{2ax+3x^2} : \sqrt[3]{ax^2+x^3}$, the limit of which (when x becomes infinite) is $x : x :: BS : BD ::$

$AS : AE$; $\therefore AS = AE$. But $AT = \frac{3ax^2+3x^3}{2ax+3x^2} - x = \frac{ax^2}{2ax+3x^2}$, the limit of which (when x becomes infinite) is $\frac{a}{3} = AS$; hence, $AE = \frac{a}{3}$; take therefore $AS = \frac{a}{3}$, and $AE = \frac{a}{3}$, join SE, and produce it indefinitely, and it will be the asymptote.

Ex. 3. *Let the equation of the curve be* $ax^4 - by^4 + cxy = 0$.

When x is infinite $ax^4 - by^4 = 0$, and $x : y :: b^{\frac{1}{4}} : a^{\frac{1}{4}}$; also, $BT = \frac{y\dot{x}}{\dot{y}} =$ (at an infinite distance) $\frac{by^4}{ax^3}$, and $AT =$



$\frac{by^4}{ax^3} - x = \frac{by^4 - ax^4}{ax^3} = 0$, therefore the asymptote passes through A; hence, take $AB : BD :: b^{\frac{1}{4}} : a^{\frac{1}{4}}$, and through D draw ADE, and it will be the asymptote; for at an infinite distance $AB : BD$ as $AB : BC$, or as $x : y$, or as $b^{\frac{1}{4}} : a^{\frac{1}{4}}$, since D and C then coincide.

To DRAW TANGENTS TO SPIRALS.

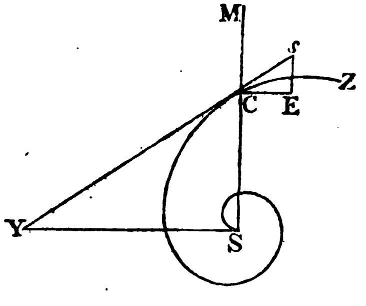
DEFINITION.

(30.) If an indefinite right line SM revolve about S, and a point C move in it continually from S, it will describe a curve called a *spiral*; *S* is called the centre, and *SC* it's ordinate.

PROP. XII.

To draw a tangent to any point C *of a spiral.*

(31.) Let *YCs* be a tangent to the spiral at *C*, and *SY* perpendicular to *SC*; draw *CE* perpendicular, and *Es* parallel to *SM*. Now the describing point *C* has two motions, one in the direction *SM*, and the other perpendicular to it, arising from the motion of *SM* about *S*. The describing point *C* is therefore under the very same circumstances as in Art. 23. upon supposition that *CE* is there perpendicular to the ordinate *CB*; the fluxions therefore must be represented here in like manner as they were there; for the fluxions at the point *C* in the directions *CE*, *CM*, and *Cs*, depend (Art. 3.) entirely upon the velocities of the describing point *C* in those directions, without any regard to what may take place afterwards from the further motion of *MS* about *S*; the fluxions therefore will be just the same as if the ordinate were moving parallel to itself, and the describing point *C* had the same two motions given to it: hence, by Art. 27. *Cs* is the fluxion of the curve, and by Art. 23. *Es* is the fluxion of the ordinate, and *CE* the fluxion in the direction perpendicular to *SC*.

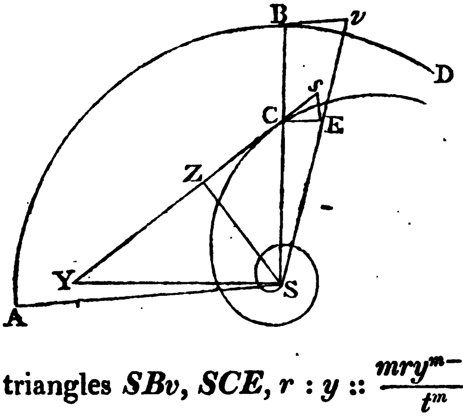


Put $SC=y$, then $Es=\dot{y}$; and, by similar triangles, ECs, CSY, Es $(\dot{y})$: CE :: $CS\,(y) : SY=\frac{y\times CE}{\dot{y}}$.

Cor. If the point C have no motion in the direction SM, the curve described will be a circle, and Es becoming $=0$, the cotemporary fluxion of a circular arc whose radius SC revolves with the same angular velocity, will be CE.

(32.) With any radius SA describe the circle ABD, produce SC to B, and SE to v meeting Bv a tangent to the circle; and suppose the angle ASC to vary as SC^m.

Put $AS=r$, $SC=y$, $AB=x$, $Bv=\dot{x}$, cotemporary with the fluxions CE, Es; for the velocity of C perpendicular to SC : velocity of B perpendicular to SB :: SC : SB; then as x is the measure of the angle ASC, let us suppose that when x becomes $=r$, y becomes t; then $x : r :: y^m : t^m$, $\therefore \frac{ry^m}{t^m}=x$, and $\frac{mry^{m-1}\dot{y}}{t^m}=\dot{x}=Bv$; and



by similar triangles SBv, SCE, $r : y :: \frac{mry^{m-1}\dot{y}}{t^m} : CE=\frac{my^m\dot{y}}{t^m}$; hence, (Art. 31.), $SY\left(=\frac{y\times CE}{\dot{y}}\right)=\frac{my^{m+1}}{t^m}$.

Cor. If SZ be perpendicular to CY, we have, by sim. triangles, YSC, SCZ, $CY : CS :: CS : CZ=$

$\frac{CS^2}{CY} = y^2 \div \sqrt{y^2 + \frac{m^2 y^{2m+2}}{t^{2m}}} = \frac{t^m y}{\sqrt{t^{2m} + m^2 y^{2m}}}$. Also, $CY : SY :: CS : SZ = \frac{SY \times CS}{CY} = \frac{m y^{m+1}}{\sqrt{t^{2m} + m^2 y^{2m}}}$.

EXAMPLES.

Ex. 1. *Let the curve be the spiral of* Archimedes.

Here $m = 1$, and $SY = \frac{y^2}{t}$; hence, $CY = \sqrt{\frac{y^4}{t^2} + y^2} = \frac{y\sqrt{y^2 + t^2}}{t}$; therefore $CZ = \frac{ty}{\sqrt{y^2 + t^2}}$. Hence also, $SZ = \frac{y^2}{\sqrt{y^2 + t^2}}$.

Ex. 2. *Let the curve be the* reciprocal *or* hyperbolic *spiral.*

Here $m = -1$, and $SY = -t$, a constant quantity. Here A lies on the other side of B, and when $x = 0$, y becomes infinite and an asymptote to the curve, for $xy = rt$ a constant quantity.

Ex. 3. *Let the spiral be the* Lituus.

Here $m = -2$, and $SY = -\frac{2t^2}{y}$.

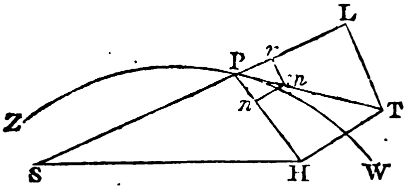
Ex. 4. *Let the curve be the* logarithmic *spiral.*

This curve is generated by the *uniform* angular motion of SC about S, whilst C recedes from S with a velocity proportional to SC; hence, sE, the fluxion of SC, varies as SC; but as the angle CSE is always the same in the same time, SC will vary as CE; hence, $CE : Es\ (\dot{y}) :: a : 1$, a constant ratio, $\therefore \frac{CE}{\dot{y}} = a$, and $SY = \frac{y \times CE}{\dot{y}} = ay$; consequently $SY : SC :: ay : y :: a : 1$, a constant ratio; hence, the triangle SCY

continues always similar to itself, and therefore the angle *SCY* is constant, and is known from the ratio of $a : 1$.

Prop. XIII.

To draw a tangent to a curve ZPW, *the nature of which is expressed in terms of* SP, HP, *drawn from two given points* S, H.



(33.) Let *PT* be the tangent at *P*, produce *SP*, and taking *Pm* to express the fluxion of the curve, if *mr* be drawn perpendicular to *PL*, and *mn* to *HP*, then (Art. 31.) *Pr* and *Pn* express the cotemporary fluxions of *SP, HP*. Draw *HT* perpendicular to *HP*, meeting the tangent *PT* at *T*, and draw *TL* perpendicular to *PL*; then the figure *PHTL* is similar to *Pnmr*, and $Pr : Pn :: PL : PH$; if therefore *PH* represent the fluxion of *PH*, *PL* will represent the cotemporary fluxion of *SP*; putting therefore $SP=x$, $HP=y$, we have the following rule:

Put the equation of the curve into fluxions; assume $\dot{y}=y$, and find $\dot{x}$; take $PL=\dot{x}$, and perpendicular to *PL* draw *LT*, meeting a perpendicular *HT* to *HP*, in *T*, and join *PT*, and it will be a tangent.

Ex. 1. Let *ZPW* be an ellipse, whose foci are *S* and *H*, and major axis a; then $x+y=a$, and (Art. 16.) $\dot{x}-\dot{y}=0$; and assuming $\dot{y}=y$ (Art. 3. Cor. 2.), we have $\dot{x}=y$; take therefore $PL=PH$, draw *LT* perpendicular to *PL*, and *HT* to *HP*, and *PT* is a tangent.

Ex. 2. Let $x^m y^n=a$ a constant quantity; then $my^n x^{m-1}\dot{x}-nx^m y^{n-1}\dot{y}=0$; and assuming $\dot{y}=y$, we get $\dot{x} = \frac{nx}{m}$; take therefore $BL = \frac{nx}{m}$, draw LT perpendicular to PT, meeting HT perpendicular to HP in T, and PT is the tangent.

Ex. 3. Let $x^m + y^n = a$ a constant quantity; then $mx^{m-1}\dot{x} - ny^{n-1}\dot{y} = 0$, and assuming $\dot{y} = y$, we get $\dot{x} = \frac{ny^n}{mx^{m-1}}$; take therefore $PL = \frac{ny^n}{mx^{m-1}}$, draw LT perpendicular to PT, meeting HT perpendicular to HP in T, and PT is the tangent.

Ex. 4. Let $x : y :: a : b$ a given ratio; then $x = \frac{ay}{b}$, and $\dot{x} = \frac{a\dot{y}}{b} =$ (by assuming $\dot{y} = y$) $\frac{ay}{b} = x$; hence, $PL = x$; take therefore $PL = PS$, draw LT perpendicular to PL, meeting HT perpendicular to HP in T, and PT is the tangent. This curve is a *circle*; see Lem. to Ex. 27. in max. and min.

ON THE BINOMIAL THEOREM.

PROP. XIV.

To express the value of $\overline{a \pm x}|^n$ *by a series.*

(34.) The square of $1+x$ is $1+2x+x^2$; the cube is $1+3x+3x^2+x^3$; &c. hence it appears, that the coefficients do not depend upon the value of x, but upon the *index* of the power; therefore if x be diminished and at last vanish, it will make no alteration in the coefficients. And as by the continual multiplication of $1+x$, we manifestly get a quantity with all the powers of x regularly ascending, let us assume $\overline{1+x}|^n = 1 + ax + bx^2 + cx^3 + dx^4 + \&c.$ n being a positive whole number. Also, if n be a negative whole number, $\overline{1+x}|^{-n}$, or

$\frac{1}{\overline{1+x}|^{n}}$ is found by division to give a series of the same kind. And if n be a fraction $\frac{r}{s}$ (r and s being whole numbers) the series will still be of the same form, as may be thus shown. The value of $\overline{1+x}|^{r}$ is expressed by $1+ax+bx^2+cx^3+$&c. but $\overline{1+x}|^{r}$ is the s^{th} power of $\overline{1+x}|^{\frac{r}{s}}$; therefore such a series must be assumed for $\overline{1+x}|^{\frac{r}{s}}$ that the s^{th} power thereof may give a series of the form $1+ax+bx^2+cx^3+$&c. Now the s^{th} power of $1+px+qx^2+rx^3+$&c. gives a series of the form $1+ax+bx^2+cx^3+$&c. therefore we must assume a series of the form $1+px+qx^2+rx^3+$&c. to represent $\overline{1+x}|^{\frac{r}{s}}$. Assume therefore in general $\overline{1+x}|^{n}=1+ax+bx^2+cx^3+$&c. Now to determine the values of a, b, c, d, &c. take the fluxion of both sides of this equation, omitting $\dot{x}$ as it will be common to every term; then take the fluxion of the resulting equation, and so on continually, and we get the following equations.

$$n\times\overline{1+x}|^{n-1}=a+2bx+3cx^2+4dx^3+\text{\&c.}$$
$$n.(n-1)\times\overline{1+x}|^{n-2}=2b+2.3cx+3.4dx^2+\text{\&c.}$$
$$n.(n-1).(n-2)\times\overline{1+x}|^{n-3}=2.3c+2.3.4dx+\text{\&c.}$$
$$\text{\&c.} \qquad\qquad \text{\&c.}$$

Now make $x=0$, and from the first equation, $n=a$; from the second, $n.(n-1)=2b$; from the third, $n.(n-1).(n-2)=2.3c$, &c. hence, $a=n$; $b=n.\frac{n-1}{2}$; $c=n.\frac{n-1}{2}.\frac{n-2}{3}$, &c. where the law of continuation is manifest. Hence, $\overline{1+x}|^{n}=1+nx+n.\frac{n-1}{2}x^2+n.\frac{n-1}{2}.\frac{n-2}{3}x^3+$&c. Now if n be a whole positive number, it is manifest that this series will terminate, for at length we must come to the coefficient $n.\frac{n-1}{2}\ldots\frac{n-n}{n+1}=0$. If n be a negative whole number, the series will never terminate, because the factors n, $n-1$, $n-2$, &c. in the numerators then become $-n$, $-n-1$, $-n-2$, &c. and therefore no one of the factors can ever become $=0$. Also, if n be a fraction, it is manifest that n, $n-1$, $n-2$, &c. can never become $=0$, because a fraction can never be destroyed by the subtraction of a whole number from it. Hence, the series will always run on *ad infinitum*, unless n be a whole positive number. If the binomial be $1-x$, then x becoming negative, the odd powers of x will be negative, and the even powers will be positive; hence, $\overline{1-x}|^n = 1 - nx + n.\frac{n-1}{2}x^2 - n.\frac{n-1}{2}.\frac{n-2}{3}x^3 + \&c.$

(35.) Hence, we may expand $\overline{a+x}|^n$. For as $a+x=a\times\left(1+\frac{x}{a}\right)$, $\therefore \overline{a+x}|^n = a^n \times \overline{1+\frac{x}{a}}|^n =$ (by writing $\frac{x}{a}$ for x in the series in the last article) $a^n \times \left(1+n.\frac{x}{a}+n.\frac{n-1}{2}.\frac{x^2}{a^2}+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{x^3}{a^3}+\&c.\right) = a^n + na^{n-1}x+n.\frac{n-1}{2}a^{n-2}x^2+n.\frac{n-1}{2}.\frac{n-2}{3}a^{n-3}x^3+\&c.$

For the different cases where the series converges or diverges, or becomes $=0$, see Dr. WARING's *Med. Anal.* p. 415.

The principal use of this rule is to extract the roots of binomials; for if n be a fraction, the series gives that root of the binomial which the fraction expresses. If therefore $n=\frac{r}{s}$, $\overline{1\pm x}|^{\frac{r}{s}} = 1 \pm \frac{r}{s}x + \frac{r}{s}\times\frac{r-s}{2s}x^2 \pm \frac{r}{s}\times\frac{r-s}{2s}\times\frac{r-2s}{3s}x^3 + \&c.$ and this form is most convenient when the index is a fraction.

EXAMPLES.

Ex. 1. *To resolve* $\frac{1}{a^2+2ax+x^2}$ *into an infinite series.*

This quantity is $\frac{1}{\overline{a+x}|^2}=\overline{a+x}|^{-2}$; which compared with $\overline{a+x}|^n$, gives $n=-2$; hence, $\overline{a+x}|^{-2}=a^{-2}-2a^{-3}x-2.\frac{-2-1}{2}.a^{-4}x^2-2.\frac{-2-1}{2}.\frac{-2-2}{3}.a^{-5}x^3-$ &c. $=\frac{1}{a^2}-\frac{2x}{a^3}+\frac{3x^2}{a^4}-\frac{4x^3}{a^5}+$ &c.

Ex. 2. *What is the value of* $\frac{1}{2az+z^2}$ *in an infinite series?*

This quantity is equal to $\frac{1}{2az\times\left(1+\frac{z}{2a}\right)}=\frac{1}{2az}\times\overline{1+\frac{z}{2a}}|^{-1}$; and by comparing $\overline{1+\frac{z}{2a}}|^{-1}$ with $\overline{1+x}|^n$, we have $x=\frac{z}{2a}$, $n=-1$; hence, $\frac{1}{2az}\times\overline{1+\frac{z}{2a}}|^{-1}=\frac{1}{2az}\times\left(1-1.\frac{z}{2a}-1.\frac{-1-1}{2}.\frac{z^2}{4a^2}\right)-$ &c. $=\frac{1}{2az}-\frac{1}{4a^2}+\frac{z}{8a^3}-$ &c.

Ex. 3. *What is the square root of* a^2+z^2, *or the value of* $\overline{a^2+z^2}|^{\frac{1}{2}}$ *in a series?*

Here $\overline{a^2+z^2}|^{\frac{1}{2}}=a\times\overline{1+\frac{z^2}{a^2}}|^{\frac{1}{2}}$; and comparing $\overline{1+\frac{z^2}{a^2}}|^{\frac{1}{2}}$ with $\overline{1+x}|^{\frac{r}{s}}$, we have $x=\frac{z^2}{a^2}$, $r=1$, $s=2$; hence, $a\times$

$$\overline{1+\frac{z^2}{a^2}}\Big|^{\frac{1}{2}}=a\times\Big(1+\frac{1}{2}\frac{z^2}{a^2}+\frac{1}{2}\times\frac{-1}{4}\frac{z^4}{a^4}+\frac{1}{2}\times\frac{-1}{4}\times\frac{-3}{6}\frac{z^6}{a^6}$$
$$+\&c.\Big)=a+\frac{1}{2}\frac{z^2}{a}-\frac{1}{8}\frac{z^4}{a^3}+\frac{1}{16}\frac{z^6}{a^5}-\&c.$$

Ex. 4. *What is the value of* $\overline{1-x}\big|^{\frac{1}{4}}$?

Here $r=1$, $s=4$, and $\overline{1-x}\big|^{\frac{1}{4}}=1-\frac{1}{4}x+\frac{1}{4}\times\frac{-3}{8}x^2-\frac{1}{4}\times\frac{-3}{8}\times\frac{-7}{12}x^3+\&c.=1-\frac{1}{4}x-\frac{3}{32}x^2-\frac{7}{128}x^3-\&c.$

Ex. 5. *What is the value of* $\overline{a-z}\big|^{\frac{1}{3}}$?

Here $\overline{a-z}\big|^{\frac{1}{3}}=a^{\frac{1}{3}}\times\overline{1-\frac{z}{a}}\Big|^{\frac{1}{3}}$, and comparing $\overline{1-\frac{z}{a}}\Big|^{\frac{1}{3}}$ with $\overline{1-x}\big|^{\frac{r}{s}}$, we have $r=1$, $s=3$, $\frac{z}{a}=x$; hence, $a\times$

$$\overline{1-\frac{z}{a}}\Big|^{\frac{1}{3}}=a\times\Big(1-\frac{1}{3}\frac{z}{a}+\frac{1}{3}\times\frac{-2}{6}\frac{z^2}{a^2}-\frac{1}{3}\times\frac{-2}{6}\times\frac{-5}{9}\frac{z^3}{a^3}$$
$$+\&c.\Big)=a-\frac{1}{3}z-\frac{1}{9}\frac{z^2}{a}-\frac{5}{81}\frac{z^3}{a^2}-\&c.$$

Ex. 6. *What is the value of* $\frac{1}{\sqrt{az-z^2}}$?

Here, $\frac{1}{\sqrt{az-z^2}}=\frac{1}{a^{\frac{1}{2}}z^{\frac{1}{2}}\times\sqrt{1-\frac{z}{a}}}=\frac{1}{a^{\frac{1}{2}}z^{\frac{1}{2}}}\times\overline{1-\frac{z}{a}}\Big|^{-\frac{1}{2}}$;

and comparing $\overline{1-\frac{z}{a}}\Big|^{-\frac{1}{2}}$ with $\overline{1-x}\big|^{\frac{r}{s}}$, we have $x=\frac{z}{a}$, $r=-1$, $s=2$; hence, $\frac{1}{a^{\frac{1}{2}}z^{\frac{1}{2}}}\times\overline{1-\frac{z}{a}}\Big|^{-\frac{1}{2}}=\frac{1}{a^{\frac{1}{2}}z^{\frac{1}{2}}}\times\Big(1-$

$$\frac{-1}{2}\,\frac{z}{a}+\frac{-1}{2}\times\frac{-3}{4}\,\frac{z^2}{a^2}-\frac{-1}{2}\times\frac{-3}{4}\times\frac{-5}{6}\,\frac{z^3}{a^3}+\&c.\Big)=$$

$$\frac{1}{a^{\frac{1}{2}}z^{\frac{1}{2}}}+\frac{1}{2}\,\frac{z^{\frac{1}{2}}}{a^{\frac{3}{2}}}+\frac{3}{8}\,\frac{z^{\frac{3}{2}}}{a^{\frac{5}{2}}}+\frac{5}{16}\,\frac{z^{\frac{5}{2}}}{a^{\frac{7}{2}}}+\&c.$$

In like manner we may raise a multinomial to any power. For let $\overline{a+bx+cx^2+dx^3+\&c.}\rvert^n=A+Bx+Cx^2+Dx^3+\&c.$ then when $x=0$, $A=a^n$. Take the fluxion, divide by $\dot{x}$, and make $x=0$, and we get $B=na^{n-1}b$. Take the fluxion again, divide by $2\dot{x}$, and make $x=0$, and we get $C=n.\frac{n-1}{2}a^{n-2}b^2+na^{n-1}c$; and thus we may proceed to get the other coefficients.

SECT. III.

ON THE METHOD OF FINDING FLUENTS.

(36.) THE business of the *direct* method of fluxions, is to find the fluxion from the fluent; to find the fluent from the fluxion is sometimes called the *inverse* method of fluxions. It is not difficult to put any quantity into fluxions, there being direct rules for that purpose; but there are no direct general rules for finding a fluent from a fluxion; and very often it is impossible to do it, except by an approximation by an infinite series, as the fluxion may be such as could not arise from putting any fluent into fluxions. We cannot therefore lay down rules for finding the fluents of any other fluxions than those whose forms show them to have been derived from some fluent.

PROP. XV.

To find the fluent of any power of a simple quantity multiplied by the fluxion of that quantity.

(37.) The fluxion of x^3 is $3\,x^2\dot{x}$, therefore we know that the fluent of $3\,x^2\dot{x}$ is x^3, and it is deduced from the fluxion, by the converse of the rule for putting x^3 into fluxions. In general, the fluxion of x^n is (Art. 12.) $nx^{n-1}\dot{x}$; therefore the fluent of $nx^{n-1}\dot{x}$ must be x^n, and this fluent is deduced from the fluxion by the following

RULE.

Add unity to the index, divide by the index so increased, and also by the fluxion of the root.

EXAMPLES.

Ex. 1. The fluent of $7x^6\dot{x}$ is x^7.

Ex. 2. The fluent of $x^9\dot{x}$ is $\frac{x^{10}}{10}$.

Ex. 3. The fluent of $5x^3\dot{x}$ is $\frac{5x^4}{4}$.

Ex. 4. The fluent of $\frac{7}{9}x^{\frac{5}{3}}\dot{x}$ is $\frac{3}{8} \times \frac{7}{9} \times x^{\frac{8}{3}} = \frac{7}{24}x^{\frac{8}{3}}$.

Ex. 5. The fluent of $\frac{6\dot{x}}{x^9}$ or $6x^{-9}\dot{x}$ is $\frac{6x^{-8}}{-8} = -\frac{3}{4x^8}$.

Ex. 6. The fluent of $\frac{3\dot{y}}{y^{\frac{3}{5}}}$ or $3y^{-\frac{3}{5}}\dot{y}$ is $\frac{5}{2} \times 3y^{\frac{2}{5}} = \frac{15}{2}y^{\frac{2}{5}}$.

If $n=0$, or the index of x be -1, the fluxion is $\frac{\dot{x}}{x}$; but this fluxion cannot be generated by x^0, because (by the Principles of Algebra) $x^0 = 1$, a constant quantity; hence, the fluent of $\frac{\dot{x}}{x}$ simply considered, cannot be found by this rule; see Art. 11. and 47.

(38.) Take the fluxion of $\frac{x^4}{x^{4+a}}$, then the dimension of x in the denominator exceeds that in the numerator by $a+1$. Hence, if a fluxion be a rational function of x multiplied into $\dot{x}$, and the greatest dimension of x in the denominator exceeds that in the numerator by unity, the fluent cannot be found in finite algebraic terms.

Ex. Let the fluxion be $\frac{ax+\overline{b+cx^3}|^{\frac{1}{2}}}{m+\overline{n+px^5}|^{\frac{1}{2}}} \times \dot{x}$, then the highest dimension of x in the denominator is $\frac{5}{2}$, and in the numerator $\frac{3}{2}$; hence the fluent cannot be found.

Prop. XVI.

To find the fluent of a binomial quantity (one part of which is constant and the other part variable) raised to a power, where the term without the vinculum is the fluxion of the variable term under the vinculum, or in a given ratio to it.

(39.) The fluxion of $\overline{a^r+x^r}|^n$ is (Cor. Art. 12.) $n \times \overline{a^r+x^r}|^{n-1} \times rx^{r-1}\dot{x}$, which is found by the same rule as the fluxion of x^n. Every complete fluxion therefore of this kind must necessarily have the index of the variable quantity without the vinculum, less by unity than the index under the vinculum. And if the given fluxion be $d \times \overline{a^r+x^r}|^{n-1} \times rx^{r-1}\dot{x}$, the fluent will be $d \times \overline{a^r+x^r}|^n$. Hence, every quantity so circumstanced, may have it's fluent found by the above rule.

If $r=1$, then $r-1=0$, and $x^0=1$; therefore the fluxion becomes $n \times \overline{a+x}|^{n-1} \times \dot{x}$.

EXAMPLES.

Ex. 1. *What is the fluent of* $\overline{a+x}|^6 \times \dot{x}$?

Here the fluxion of the root $a+x$ is $\dot{x}$; hence, the fluent is $\frac{\overline{a+x}|^7 \times \dot{x}}{7\dot{x}} = \frac{\overline{a+x}|^7}{7}$.

Ex. 2. *What is the fluent of* $\overline{a^2+x^2}|^{\frac{1}{2}} \times x\dot{x}$?

Here the fluxion of the root a^2+x^2 is $2x\dot{x}$; hence, the fluent is $\frac{\overline{a^2+x^2}|^{\frac{3}{2}} \times x\dot{x}}{\frac{3}{2} \times 2x\dot{x}} = \frac{\overline{a^2+x^2}|^{\frac{3}{2}}}{3}$.

Ex. 3. *What is the fluent of* $\overline{a^4-x^4}|^{\frac{5}{3}}\times 3x^3\dot{x}$?

Here the fluxion of the root a^4-x^4 is $-4x^3\dot{x}$; hence, the fluent is $\frac{\overline{a^4-x^4}|^{\frac{8}{3}}\times 3x^3\dot{x}}{\frac{8}{3}\times -4x^3\dot{x}}=-\frac{9\times\overline{a^4-x^4}|^{\frac{8}{3}}}{32}$.

Ex. 4. *What is the fluent of* $\frac{x^8\dot{x}}{\overline{a^9+6x^9}|^{\frac{1}{2}}}$?

This quantity is $=\overline{a^9+6x^9}|^{-\frac{1}{2}}\times x^8\dot{x}$; and the fluxion of the root a^9+6x^9 is $54x^8\dot{x}$; therefore the fluent is $\frac{\overline{a^9+6x^9}|^{\frac{1}{2}}\times x^8\dot{x}}{\frac{1}{2}\times 54x^8\dot{x}}=\frac{\overline{a^9+6x^9}|^{\frac{1}{2}}}{27}$.

Quantities which at first do not stand under this form, may frequently be reduced to it.

Ex. 5. *What is the fluent of* $\frac{a\dot{x}}{\overline{a^2+x^2}|^{\frac{3}{2}}}$?

First, $a^2+x^2=(a^2x^{-2}+1)\times x^2$; therefore $\overline{a^2+x^2}|^{\frac{3}{2}}=\overline{a^2x^{-2}+1}|^{\frac{3}{2}}\times x^3$; hence, $\frac{a\dot{x}}{\overline{a^2+x^2}|^{\frac{3}{2}}}=\frac{a\dot{x}}{\overline{a^2x^{-2}+1}|^{\frac{3}{2}}\times x^3}=\overline{a^2x^{-2}+1}|^{-\frac{3}{2}}\times ax^{-3}\dot{x}$, where the index of x without is less by unity than that under the vinculum; hence, the fluent is $\frac{\overline{a^2x^{-2}+1}|^{-\frac{1}{2}}\times ax^{-3}\dot{x}}{-\frac{1}{2}\times -2a^2x^{-3}\dot{x}}=\frac{1}{\overline{a^2x^{-2}+1}|^{\frac{1}{2}}\times a}=\frac{x}{\overline{a^2+x^2}|^{\frac{1}{2}}\times a}$.

Ex. 6. *What is the fluent of* $\frac{\dot{x}}{x^2\sqrt{a^2+x^2}}$?

This is reduced to $\frac{x^{-3}\dot{x}}{\sqrt{a^2x^{-2}+1}}=\overline{a^2x^{-2}+1}|^{-\frac{1}{2}}\times x^{-3}\dot{x}$, whose fluent is $-\frac{1}{a^2}\times\sqrt{a^2x^{-2}+1}$.

Or, in general, every fluxion of this kind $\frac{x^r\dot{x}}{\overline{a^m+x^m}|^{\frac{1}{n}}}$ can be reduced so that the fluent may be found by this Rule, if $r=\frac{m}{n}-m-1$; for by reduction it becomes

$$\frac{x^r\dot{x}}{\overline{a^m x^{-m}+1}|^{\frac{1}{n}}\times x^{\frac{m}{n}}}=x^{r-\frac{m}{n}}\dot{x}\times\overline{a^m x^{-m}+1}|^{-\frac{1}{n}},$$ so that $r-\frac{m}{n}$ must $=-m-1$, or $r=\frac{m}{n}-m-1$, in which case the fluent is $\frac{\overline{a^m x^{-m}+1}|^{1-\frac{1}{n}}}{\left(1-\frac{1}{n}\right)\times -ma^m}$.

Ex. 7. *To find the fluent of* $\frac{\dot{x}\sqrt{a^2+x^2}}{x^{2m}}=\dot{F}$, m *being a whole positive number greater than unity.*

Now $\dot{F}=\frac{\dot{x}}{x^{2m-1}}\times\frac{\sqrt{a^2+x^2}}{x}$; put $\frac{\sqrt{a^2+x^2}}{x}=y$, then $x^2=\frac{a^2}{y^2-1}$, and $\frac{1}{x^{2m-2}}=\frac{\overline{y^2-1}|^{m-1}}{a^{2m-2}}$, and $-(2m-2)\times\frac{\dot{x}}{x^{2m-1}}=\frac{(m-1)\times\overline{y^2-1}|^{m-2}\times 2y\dot{y}}{a^{2m-2}}$; hence, $\dot{F}=\frac{2\times(1-m)}{(2m-2)\times a^{2m-2}}\times\overline{y^2-1}|^{m-2}\times y^2\dot{y}$; expand $\overline{y^2-1}|^{m-2}$ by the Binomial Theorem, multiply each term by $y^2\dot{y}$, and the fluent of each term is found by Art. 37.

If $m=1$, then $\dot{F}=\frac{y\dot{y}}{y^2-1}$, and (Art. 45.) $F=\frac{1}{2}$ h. l. $(y^2-1)=$ h. l. $\sqrt{y^2-1}$.

Ex. 8. *To find the fluent of* $\dfrac{x^{\frac{1}{3}}\dot{x}}{\overline{a^2 - x^{\frac{2}{3}}}|^{\frac{3}{5}}}$.

Put $\overline{a^2 - x^{\frac{2}{3}}}|^{\frac{5}{3}} = y$; then the fluxion is transformed into $\frac{5}{2} \times \left(y^{\frac{4}{3}}\dot{y} - a^2 y^{-\frac{1}{3}}\dot{y}\right)$, and the fluent is $\frac{5}{2} \times \left(\frac{3}{7}y^{\frac{7}{3}} - \frac{3a^2}{2}.y^{\frac{2}{3}}\right)$.

(40.) If both quantities under the vinculum be variable, and the quantity without be the fluxion of the quantity under the vinculum, or in a constant ratio to it, the fluent may be found by this rule. Thus, the fluent of $\overline{a^2y^2 + y^4}|^{\frac{1}{2}} \times (2a^2y\dot{y} + 4y^3\dot{y})$ is $\frac{2}{3} \times \overline{a^2y^2 + y^4}|^{\frac{3}{2}}$; but these cases seldom occur.

Prop. XVII.

To find the fluent of $\overline{a + cz^n}|^m \times dz^{rn-1}\dot{z}$, *where the index of* z *without the vinculum increased by unity, is some multiple of the index of* z *under the vinculum.*

(41.) Put $a + cz^n = x$, then $z^n = \dfrac{x - a}{c}$, $\therefore$ $z^{rn} = \dfrac{\overline{x - a}|^r}{c^r}$, take it's fluxion, and $rnz^{rn-1}\dot{z} = \dfrac{r \times \overline{x - a}|^{r-1}\dot{x}}{c^r}$, $\therefore$ $z^{rn-1}\dot{z} = \dfrac{1}{nc^r} \times \overline{x - a}|^{r-1} \times \dot{x}$; hence (putting $r - 1 = s$), $dz^{rn-1}\dot{z} = \dfrac{d}{nc^r} \times \overline{x - a}|^s \times \dot{x} =$ (by expanding $\overline{x - a}|^s$) $\dfrac{d}{nc^r} \times \dot{x} \times \left(x^s - sax^{s-1} + s.\dfrac{s-1}{2}a^2x^{s-2} - \text{\&c.}\right)$ substitute this quantity for $dz^{rn-1}\dot{z}$, and x^m for $\overline{a+cz^n}\rvert^{m}$, and the given fluxion is transformed to $\frac{d}{nc^r} \times$ $x^m\dot{x} \times \left(x^s - sax^{s-1} + s.\frac{s-1}{2}a^2x^{s-2} - \&c.\right) = \frac{d}{nc^r} \times$ $\left(x^{m+s}\dot{x} - sax^{m+s-1}\dot{x} + s.\frac{s-1}{2}a^2x^{m+s-2}\dot{x} - \&c.\right)$ the fluent of each of which terms is found by the Rule in Art. 37. hence, the fluent required is $\frac{d}{nc^r} \times$ $\left(\frac{x^{m+s+1}}{m+s+1} - \frac{sax^{m+s}}{m+s} + \frac{s.\frac{s-1}{2}.a^2x^{m+s-1}}{m+s-1} - \&c.\right)$ Now let us consider when the fluent of the given fluxion can be expressed in finite terms.

1st. If r, and consequently s, be a whole *positive* number, the series arising from the expansion of $\overline{x-a}\rvert^{s}$ will terminate, and the fluent can always be found if m be a *positive* whole number, or a *positive* or *negative* fraction.

2dly. If r be a *positive* whole number, and m a *negative* whole number greater in magnitude than $s+1$, or r, the fluent can always be found. But if m be a *negative* whole number equal to or less in magnitude than r, the denominator of some one of the terms must become $=0$, in which case the fluent of that term fails; for in the fluxion it was of this form $x^{-1}\dot{x}$, which by Art. 38. admits of no fluent by the rule there given; it may however be found by logarithms, as will be explained in Art. 45.

3dly. The given fluxion, by reduction, becomes $\overline{az^{-n}+c}\rvert^{m} \times dz^{(m+r)+n-1}\dot{z}$; hence, if m and r be both fractions, but such that $m+r$ may be a whole *negative* number, the fluent can always be found. This will appear, by transforming the fluxion as before; and the series will always terminate; nor can any of the denominators of the terms of the fluent become equal to nothing, so as to make the fluent of such term fail, as it is here taken.

When fluents cannot be found in finite terms, instead of having recourse to infinite series, they may frequently be found in terms of circular arcs and logarithms; and these having been computed to a considerable degree of accuracy, they may be used for all practical purposes.

To FIND FLUENTS BY LOGARITHMS.

(42.) The property of logarithms, or their relation to natural numbers, as has been already explained in Algebra, is this, that as the natural numbers increase in geometric progression, their logarithms increase in arithmetic progression.

(43.) Let a increase till it becomes $b, c, \dots . m, n, o,$ &c. and suppose $a : b :: b : c ::$ &c. $:: m : n ::$ &c. then $a : m :: a-b : m-n$; now $a-b$ is the increment of a, and $m-n$ is the increment of m; hence, $a : m ::$ the increment of a : the increment of m; and as this is true in every state of the increments, if we make them vanish, we have $a : m$ in the *limiting* ratio of the increment of a : the increment of m, that is, as the fluxion of a : the fluxion of m, by Art. 7.

(44.) Let y be any number, and x it's logarithm; then if x increase uniformly, or if $\dot{x}$ be constant, y will increase in geometric progression, therefore by the last article, y varies as $\dot{y}$, and $\frac{y}{\dot{y}}$ is constant; hence, $\frac{y\dot{x}}{\dot{y}}$ is constant; put therefore $\frac{y\dot{x}}{\dot{y}} = M$, and we have $\dot{x} = M \times$

$\frac{\dot{y}}{y}$; that is, the fluxion of any logarithm is equal to a *constant* quantity multiplied into the fluxion of the number divided by the number. The quantity M is called the *modulus* of the system, and may be assumed of any value.

If $M=1$, the logarithms are called *hyperbolic*, because the same logarithms may be deduced from the hyperbola, as will appear hereafter. In this case $\dot{x}=\frac{\dot{y}}{y}$.

Prop. XVIII.

To find the fluent of a fluxion, which is the fluxion of any quantity ($\dot{y}$) *divided by that quantity* (y), *or in a given ratio to it.*

(45.) Put $x=$ the hyperbolic logarithm of y; then by Art. 44. $\frac{\dot{y}}{y}=\dot{x}$, and the fluent of $\frac{\dot{y}}{y}$* is x. And as y, although here a simple quantity, may represent any compound quantity whatever, and $\dot{y}$ it's fluxion, we have the following

RULE.

When any fluxional expression appears to be the fluxion of a quantity divided by the quantity itself, it's fluent is the hyperbolic logarithm of that quantity.

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* If $x=$ hyp. log.$-y$, then $\dot{x}=\frac{\dot{y}}{y}$; the fluent therefore of $\frac{\dot{y}}{y}$ is h. l. $\pm y$; but the negative value belongs to another system. Also, the fluent of $\frac{\dot{y}}{y}$ is h. l. ay, since the fluxion of h. l. $ay=\frac{a\dot{y}}{ay}=\frac{\dot{y}}{y}$. Hence, in all the above Examples, the fluents may be multiplied by a constant quantity: thus, the fluent of $\frac{\dot{x}}{x\pm a}$ is h. l. $m\times(x\pm a.)$

EXAMPLES.

Ex. 1. The fluent of $\frac{\dot{x}}{x \pm a}$ is the h. l. (hyperbolic logarithm) of $(x \pm a)$.

Ex. 2. The fluent of $\frac{2x\dot{x}}{a^2+x^2}$ is the h. l. (a^2+x^2).

Ex. 3. The fluent of $\frac{nx^{n-1}\dot{x}}{a^n+x^n}$ is the h. l. (a^n+x^n).

These fluents are obvious, the given fluxion being manifestly the fluxion of the quantity divided by the quantity, for the numerator is the fluxion of the denominator.

Ex. 4. The fluent of $\frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$ is the h. l. of $(x+\sqrt{x^2 \pm a^2})$.

For, put $x^2 \pm a^2 = v^2$, then $x\dot{x} = v\dot{v}$, $\therefore$ $x : v :: \dot{v} : \dot{x}$, and $x+v : v :: \dot{x}+\dot{v} : \dot{x}$; hence, $\frac{\dot{x}+\dot{v}}{x+v} = \frac{\dot{x}}{v} = \frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$; therefore the fluent of $\frac{\dot{x}+\dot{v}}{x+v}$, or of $\frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$, is the h. l. $(x+v)$ = h. l. $(x+\sqrt{x^2 \pm a^2})$.

Ex. 5. The fluent of $\frac{\dot{x}}{\sqrt{x^2 \pm 2ax}}$ is the h. l. $(x \pm a+\sqrt{x^2 \pm 2ax})$.

For, put $\sqrt{x^2 \pm 2ax} = y$, then $x^2 \pm 2ax + a^2 = y^2 + a^2$, and $x \pm a = \sqrt{y^2+a^2}$; hence, $\dot{x} = \frac{y\dot{y}}{\sqrt{y^2+a^2}}$, consequently $\frac{\dot{x}}{\sqrt{x^2 \pm 2ax}} = \frac{\dot{y}}{\sqrt{y^2+a^2}}$, whose fluent, by the last example, is h. l. $(y+\sqrt{y^2+a^2})$ = h. l. $(x \pm a+\sqrt{x^2 \pm 2ax})$.

Ex. 6. The fluent of $\frac{2a\dot{x}}{a^2-x^2}$ is the h. l. $\frac{a+x}{a-x}$.

For $\frac{2a\dot{x}}{a^2-x^2}=\frac{\dot{x}}{a+x}-\frac{-\dot{x}}{a-x}$, whose fluent is the h. l. $(a+x)$ − h. l. $(a-x)$ = h. l. $\frac{a+x}{a-x}$, as shown in the Algebra, Art. 388. In like manner, the fluent of $\frac{2a\dot{x}}{x^2-a^2}$ is h. l. $\frac{x-a}{x+a}$.

Ex. 7. The fluent of $\frac{2a\dot{x}}{x\sqrt{a^2+x^2}}$ is the h. l. $\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}$.

For, put $\sqrt{a^2+x^2}=y$, then $a^2+x^2=y^2$, therefore $x\dot{x}=y\dot{y}$, and $\frac{2a\dot{x}}{xy}=\frac{2a\dot{y}}{x^2}$; that is, $\frac{2a\dot{x}}{x\sqrt{a^2+x^2}}=\frac{2a\dot{y}}{y^2-a^2}$, whose fluent, by the last example, is h. l. $\frac{y-a}{y+a}$ = h. l. $\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}$. In like manner, the fluent of $\frac{2a\dot{x}}{x\sqrt{a^2-x^2}}$ is h. l. $\frac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}}$.

Ex. 8. The fluent of $\frac{x^{-2}\dot{x}}{\sqrt{b^2+x^{-2}}}$ is − h. l. $\frac{1+\sqrt{1+b^2x^2}}{x}$.

For, put $\frac{1}{x}=y$, then $x^{-2}\dot{x}=-\dot{y}$; hence, the fluxion becomes $\frac{-\dot{y}}{\sqrt{b^2+y^2}}$, whose fluent is (by Example 4.)

$-$ h. l. $(y+\sqrt{b^2+y^2}) = -$ h. l. $\left(\frac{1}{x}+\sqrt{b^2+\frac{1}{x^2}}\right) = -$ h. l. $\frac{1+\sqrt{1+b^2x^2}}{x}$.

In like manner, the fluent of $\frac{b^2\dot{y}}{y\sqrt{b^2-y^2}}$ is found to be h. l. $\frac{b+\sqrt{b^2-y^2}}{by}$.

These are some of the most useful forms of fluxions whose fluents may be found by a table of hyperbolic logarithms; which table may be supplied, by multiplying the logarithm found from the common tables by 2,30258509, which will give the corresponding hyperbolic logarithm.

Ex. The fluent of $\frac{\dot{x}}{1+x}$ is the h. l. of $(1+x)$; if $x=1$, the fluent is the h. l. of $2=0,693147$; if $x=4$, the fluent is the h. l. of $5=1,6094379$.

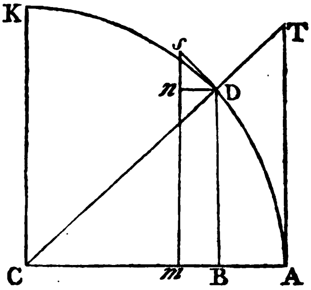
To find FLUENTS by CIRCULAR ARCS.

Prop. XIX.

The length of a circular arc for every degree, minute, and second, to radius $= 1$, being given, to find from thence certain fluents.

(46.) Let *AD* be a circular arc whose centre is *C*, *AT* it's tangent, *DB* it's sine; draw *ms* parallel to *BD* meeting the tangent *Ds* in *s*, and *Dn* parallel to *Bm*.

Put $CD=a$, $AB=x$, $BD=y$, $AD=z$, $AT=t$, $CT=s$; then by Art. 23. $Ds=\dot{z}$, $Dn=\dot{x}$, $ns=\dot{y}$. Now the triangles *CBD*, *snD* are similar, for they are right angled at *B* and *n*, and the angle *sDn* = *CDB*, because *nDC* is the complement of each. Hence, $y : a :: \dot{x} : \dot{z} = \frac{a\dot{x}}{y}$; but $y = \sqrt{CD^2 - CB^2} = \sqrt{a^2 - \overline{a-x}|^2} = \sqrt{2ax - x^2}$; $\therefore \dot{z} = \frac{a\dot{x}}{\sqrt{2ax-x^2}}$. Also, $\sqrt{a^2-y^2}$ (*BC*) $: a :: \dot{y} : \dot{z} = \frac{a\dot{y}}{\sqrt{a^2-y^2}}$. Again, by sim. triangles *CAT*, *CBD*, s (*CT*) : a (*CA*) :: a (*CD*) : $CB = \frac{a^2}{s}$, $\therefore AB = a - \frac{a^2}{s}$, whose fluxion *Bm* or $Dn = \frac{a^2\dot{s}}{s^2}$; hence, from the sim. trian. *Dsn*, *CAT*, $\sqrt{s^2-a^2}$ (*AT*) $: s :: \frac{a^2\dot{s}}{s^2} : \dot{z} = \frac{a^2\dot{s}}{s\sqrt{s^2-a^2}}$. Lastly, $\sqrt{s^2-a^2} = t$, $\therefore \frac{s\dot{s}}{\sqrt{s^2-a^2}} = \dot{t}$, and $\dot{z} \left(= \frac{a^2\dot{s}}{s\sqrt{s^2-a^2}}\right) = \frac{a^2\dot{t}}{s^2} = \frac{a^2\dot{t}}{a^2+t^2}$. Hence, the fluxion of the arc *AD*, or $\dot{z}$, is expressed under four different forms in terms of the right sine, versed sine, tangent and secant; consequently the fluent of each of these fluxions will be expressed by z. Hence,



1st Fluent of $\frac{a\dot{y}}{\sqrt{a^2-y^2}}$ is a circ. arc whose rad. is a and sine y.

2d Fluent of $\frac{a\dot{x}}{\sqrt{2ax-x^2}}$ is a circ. arc whose rad. is a and versed sine x.

3d Fluent of $\frac{a^2\dot{t}}{a^2+t^2}$ is a circ. arc whose rad. is a and tangent t.

4th Fluent of $\frac{a^2\dot{s}}{s\sqrt{s^2-a^2}}$ is a circ. arc whose rad. is a and secant s.

Now by a table exhibiting the length of circular arcs for all degrees, &c. of the quadrant to radius unity, if these arcs be multiplied by a, we shall have their lengths to the radius a. Hence, for example, what is the fluent of $\frac{a\dot{y}}{\sqrt{a^2-y^2}}$, when y is the sine of 30° ? The length of an arc of 30° to radius 1, is 0,5235987 : hence, the length of the arc to radius a, is $a \times 0{,}5235987$, the fluent required. Thus, the fluents of all fluxions under any of these forms may be found.

If the fluxions had been $\frac{\dot{y}}{\sqrt{a^2-y^2}}$, $\frac{\dot{x}}{\sqrt{2ax-x^2}}$, $\frac{a\dot{t}}{a^2+t^2}$, $\frac{a\dot{s}}{s\sqrt{s^2-a^2}}$, that is, a times less than those stated above, the fluents would have been a times less; that is, the circular arcs would have been reduced to a radius$=1$, and the respective sines, versed sines, tangents and secants, would have been $\frac{y}{a}$, $\frac{x}{a}$, $\frac{t}{a}$, $\frac{s}{a}$.

If in the expression $\frac{a^2\dot{t}}{a^2+t^2}$ the fluxion ($\dot{z}$) of the circular arc, we suppose $v=a\times\frac{a+t\sqrt{-1}}{a-t\sqrt{-1}}$, we have $\dot{z}=\frac{a\dot{v}}{2v\sqrt{-1}}$, and $z=\frac{a}{2c\sqrt{-1}}\times$ h. l. v (Note to

Art. 45, c being a constant quantity), where the imaginary value v is destroyed by the imaginary quantity $c\sqrt{-1}$, and the quantity expresses accurately the value of z. In this light we are to understand all real quantities under the form of imaginary ones, such imaginary quantities destroying each other. Thus, when the roots of equations are imaginary, the imaginary parts destroy each other, and appear not in the equation itself.

(47.) A fluent can have but one fluxion, but a fluxion may have an infinite number of fluents; thus, the fluent of $\dot{x}$ is x, or $x \pm a$, whatever be the value of the constant part a. By Prop. 4. in taking the fluxion of a binomial, the constant part goes out, and therefore when the fluent is taken back again, that constant part does not appear. Now to determine, in any particular case, what this constant part is to be, or whether any such quantity is to be annexed, consider whether the fluent first taken becomes equal to nothing, or of a known value, at the time it ought; if it do, it requires no constant quantity to be added; if it do not, such a quantity must be annexed to it, as will make it become equal to nothing, or to it's proper value. This is called the *correction* of a fluent.

It is here necessary to be observed, that the correction must be made to the fluent as it is first found, that is, before any reductions take place, such as raising the quantity to any power, extracting the roots, &c. since new values may be so introduced. If $\dot{x}=\dot{z}$, then $x=z+Cor.$ in general. We must not say $x=z$, and $x^2=z^2$ and then apply the correction, since $x=z$ is not a true equation when a correction is necessary. If when $x=0$, $z=a$, then $x=z-a$ is the true equation. But in the equation $x^2=z^2+Cor.$ it would be $x^2=z^2-a^2$, and $x=\sqrt{z^2-a^2}$ which is different from the former value of x corrected. When we assume $x^2=z^2$, then $x=\pm z$ and $\dot{x}=\pm\dot{z}$ which is not our original equation.

Also, from $x^2 = z^2 - a^2$ we have $2x\dot{x} = 2z\dot{z}$, which is not our original equation unless $x = z$ which is contrary to our supposition, since when $x = 0$, $z = a$.

Further, the fluent of $x^n\dot{x}$ is $\frac{x^{n+1}}{n+1}$, and if $n = -1$, this becomes $\frac{x^0}{0} = \frac{1}{0} =$ infinity, and is the same whatever be the value of x. Hence, if $x^n\dot{x}$ express the fluxion of any quantity, the fluent when $n = -1$ must want a correction, since the value of the fluent, depending upon x, cannot be the same for all values of x. Let therefore the fluent become $= 0$ when $x = a$, and the correct fluent is $\frac{x^{n+1} - a^{n+1}}{n+1}$. Now x remaining constant, let n vary till it becomes $= -1$, and in this case the numerator and denominator vanish together; hence, (Note to Art. 7.) the value of the fraction at that time = the fluxion of the numerator divided by the fluxion of the denominator = (Prop. 51.) $\frac{x^{n+1}\dot{n} \times \text{h. l. } x - a^{n+1}\dot{n} \times \text{h. l. } a}{\dot{n}}$ (when $n = -1$) h. l. $x -$ h. l. $a =$ h. l. $\frac{x}{a}$. Hence, in this, and all other like cases, it is not correct to say that the fluent found by the common rule fails, for it still exhibits the true fluent, but under such circumstances as requires a further reduction in order to determine it's value. The fluent comes out the same as if you had at first taken $n = -1$, and found the fluent by logarithms. The fluent therefore is general; and if any calculations be carried on from this point, the conclusion will still be true when you make $n = -1$.

Or if we want to find the value of $\frac{x^{n+1}}{d^{n+1}}$ found from the fluent of $x^{n+1}\dot{x}$ when $n = -1$, we must get the ultimate value, making n only variable till it becomes

$= -1$; hence, the ultimate value is $\frac{x^{n+1}\dot{n} \times \text{h. l.}\, x}{d^{n+1}\dot{n} \times \text{h. l.}\, d} = \frac{\text{h. l.}\, x}{\text{h. l.}\, d}$.

(48.) Although the fluxion of a quantity be *relative*, that is, if $\dot{x}$ denote the fluxion of x, then will $nx^{n-1}\dot{x}$ be the fluxion of x^n, where $\dot{x}$ may be assumed of any magnitude, yet the fluents are not at all affected by varying $\dot{x}$, the fluents of these quantities $\dot{x}$ and $nx^{n-1}\dot{x}$ being x and x^n, whatever be the value of $\dot{x}$. Hence, of whatever magnitude we assume the fluxion of any quantity, the fluent will always give the quantity generated. In the following Problems, therefore, the fluxion of the area, solid, curve line or surface, may be assumed of any magnitude, and the fluent, corrected if necessary, will give the quantity which has been generated.

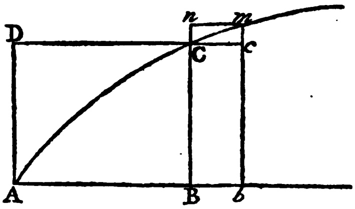
SECT. IV.

TO FIND THE AREAS OF CURVES.

PROP. XX.

To find the area ABC *of any curve, whose ordinate* BC *is perpendicular to the abscissa* AB.

(49.) LET *ABC* be any curvilinear area generated by the uniform motion of the ordinate *BC*; on *AB*, *BC* describe the parallelogram *ABCD*, and conceive this to have been generated by



the same uniform motion of a line equal and parallel to *AD*; draw *bm* parallel to *BC*, and complete the parallelogram *Bbmn*, and produce *DC* to *c*. Then *AD* being constant whilst *BC* varies, the next increment of the parallelogram is *BCcb*, and the cotemporary increment of the area *ABC* is *BCmb*; hence, the ratio of the increment *BCcb* of the parallelogram to the cotemporary increment *BCmb* of the area *ABC*, is always nearer to a ratio of equality, than *BCcb* : *Bnmb*, or nearer than *BC* : *bm*; now let *bm* move up to, and coincide with *BC*, in order to obtain the *limiting* ratio of the increments, and we get the *limiting* ratio of *BC* : *bm*, a ratio of equality; hence, *a fortiori*, the *limiting* ratio of the increment *BCcb* of the parallelogram, to the cotemporary increment *BCmb* of the area *ABC*, is a ratio of equality; therefore by Prop. 2. the fluxion of the parallelogram *ABCD* is equal to the fluxion of the area *ABC*; but *BCcb* being the increment of the parallelogram *uniformly* generated, will represent it's fluxion, by Prop. 1. hence, the fluxion of the area of the curve *ABC* will be represented by *BCcb*, the cotemporary fluxion of the abscissa *AB* being *Bb*. If therefore $AB = x$, $BC = y$, $Bb = \dot{x}$, and $A =$ the area *ABC*, then will $\dot{A} = BCcb = y\dot{x}$; the fluent of which, corrected if necessary, gives A.

COR. 1. If the ordinate be inclined to the axis at an angle whose sine is s, the fluxion of the area is $sy\dot{x}$.

COR. 2. Hence, the fluxion of any area, generated by the motion of a straight line in a direction perpendicular to itself, is as the length of the generating line and it's velocity conjointly. And as a curve line, moving in a direction perpendicular to itself, must describe the same area as a straight line of the same length moving with the same velocity, the fluxion of the surface generated by a curve line, so moving, must be as it's length and velocity conjointly.

EXAMPLES.

Ex. 1. *Let* AC *be any* parabola; *to find it's area.*

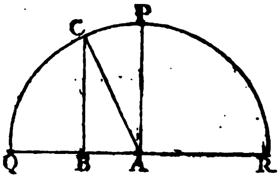
Here $ax = y^n$; hence, $a\dot{x} = ny^{n-1}\dot{y}$, and $\dot{x} = \frac{ny^{n-1}\dot{y}}{a}$, $\therefore y\dot{x} = \frac{ny^n\dot{y}}{a} = \dot{A}$, whose fluent (Art. 37.) $A = \frac{ny^{n+1}}{(n+1) \times a}$ $+ C$ (C being the correction if necessary) $= \frac{n}{n+1} \times \frac{y^n}{a}$ $\times y + C = \frac{n}{n+1} \times xy + C$; now when $A = 0$, $x = 0$, $\therefore$ $C = 0$; hence, $A = \frac{n}{n+1} \times xy$.

If $n=2$, it becomes the common parabola, and the area $=\frac{2}{3}xy$.

If $n=1$, * the figure becomes a triangle, and the area$=\frac{1}{2}xy$.

Ex. 2. *To find the area of a circle, whose radius is unity.*

Let A be the centre of the circle; draw BC, AP,



perpendicular to QR; and join AC. Put $AC=1$, $AB=x$, $BC=y$; then $x^2+y^2=1$, $\therefore y=\overline{1-x^2}|^{\frac{1}{2}}=1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-\frac{5x^8}{128}-$&c. (Art. 34.); $\therefore \dot{A}=y\dot{x}=\dot{x}-\frac{x^2\dot{x}}{2}-\frac{x^4\dot{x}}{8}-\frac{x^6\dot{x}}{16}-\frac{5x^8\dot{x}}{128}-$&c. the fluxion of the area $BAPC$ whose fluent is $A=x-\frac{x^3}{6}-\frac{x^5}{40}-\frac{x^7}{112}-\frac{5x^9}{1152}-$&c. $+C$; now when $x=0$, $A=0$, $\therefore C=0$; hence, $A=x-\frac{x^3}{6}-\frac{x^5}{40}-\frac{x^7}{112}-\frac{5x^9}{1152}-$&c. Now if the arc $PC=30°$, $x=\frac{1}{2}$; and the area $ABCP=5-0{,}0208333-0{,}0007812-0{,}0000698-0{,}0000085-0{,}0000012-$&c.$=0{,}4783055$. But as $x=\frac{1}{2}$, $y=\sqrt{\frac{3}{4}}$; therefore the area of the triangle $ACB=\frac{1}{4}\times\sqrt{\frac{3}{4}}=0{,}2165063$, which subtracted from 0,4783055 leaves 0,2617992 the area of the sector ACP; which multiplied by 12 gives 3,14159 &c.$=$the area of the whole circle.

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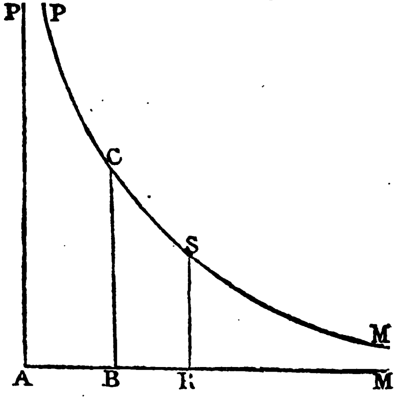
* If $n=1$, $ax=y$, and $x:y::1:a$, that is, in a constant ratio, which is the case when AC is a straight line, because the triangle ABC continues always similar to itself.

COR. If r = radius of any circle, a = it's area; then, since circles vary as the squares of their radii, $1^2 : r^2 :: 3{,}14159$ &c. : $a = 3{,}14159$ &c. $\times r^2$. If d = the diameter, then $r = \frac{d}{2}$, and $r^2 = \frac{d^2}{4}$; hence, $a = 3{,}14159$ &c. $\times \frac{d^2}{4} = 0{,}78539$ &c. $\times d^2$.

EX. 3. *To find the area of an* hyperbola *between the asymptotes* AP, AM, *and the curve* MP.

Put $AB = x$, $BC = y$; then $y = \frac{1}{x^n}$, and the fluxion of the area $APCB = y\dot{x} = \frac{\dot{x}}{x^n} = x^{-n}\dot{x} = \dot{A}$, whose fluent



is $A = \frac{x^{1-n}}{1-n} + C$.

CASE 1. If n be *less* than unity, when $A = 0$, $x = 0$, $\therefore \frac{x^{1-n}}{1-n} = 0$; hence, $C = 0$; therefore the area $APCB$ (infinite in extent) $= \frac{x^{1-n}}{1-n}$, a finite quantity when x is finite.

CASE 2. If n be *greater* than unity, the index $1-n$ being negative, x must come into the denominator, and the fluent will become $A = \frac{1}{(1-n) \times x^{n-1}} + C = -\frac{1}{(n-1) \times x^{n-1}} + C$; now when $A=0$, $x=0$, consequently $C = \frac{1}{(n-1) \times x^{n-1}}$ is *infinite*, because the denominator becomes $=0$; therefore the area $APCB = \frac{1}{(1-n) \times x^{n-1}} + C$ is *infinite*. Whenever there is a negative index, the quantity must always be transferred from the numerator to the denominator, or the contrary, before it's value, in any particular case, can be found.

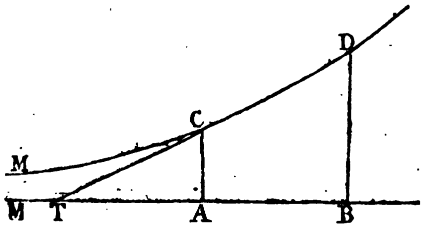
CASE 3. In respect to the area *BCM*, as this area decreases by the same quantity that *ABCP* increases, it will have the same fluxion, only with a contrary sign, by Art. 16. hence, the fluent will be the same with the sign changed, that is, $BCM = \frac{x^{1-n}}{n-1} + C$. If n be *greater* than unity, $BCM = \frac{1}{(n-1).x^{n-1}} + C$; and when x is infinite, $BCM=0$; hence, $0 = \frac{1}{(n-1).x^{n-1}} + C$, and therefore $C = -\frac{1}{(n-1).x^{n-1}} = 0$, x being infinite; consequently $BCM = \frac{1}{(n-1).x^{n-1}}$.

CASE 4. If n be *less* than unity, and x become infinite, $C = \frac{x^{1-n}}{1-n}$, an *infinite* quantity; hence, the area $BCM = \frac{x^{1-n}}{n-1} + C$ is *infinite*.

CASE 5. If $n=1$, the hyperbola becomes the common hyperbola. Let $AB = BC = 1$, $BR = x$, $RS = y$, then $AR = 1 + x$, and $y = \frac{1}{1+x}$, therefore the fluxion of the area $BCSR = \frac{\dot{x}}{1+x}$, whose fluent, by Art. 45. is the h. l. $(1+x)$, which wants no correction, because when $x = 0$, the area $BCRS = 0$, and the fluent becomes the h. l. 1 which $= 0$. Hence it appears, that any area $BCSR$ is the h. l. of the abscissa AR, and that the whole area BCM is infinite. The *modulus* is here unity.

Ex. 4. *Let* MCD *be the* logarithmic *curve; to find it's area.*

The property of the logarithmic curve is this, that if the abscissa AB increase in arithmetical progression, the ordinate BD will increase in geometrical progression;

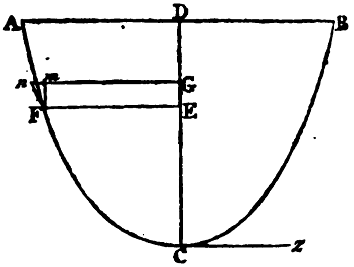


$\therefore$ if $x = AB$, $y = BD$, $a = AC$, then (Art. 44.) $M = \frac{y\dot{x}}{\dot{y}}$, which (by Art. 23.) is the sub-tangent AT; hence, $\dot{A} = y\dot{x} = M\dot{y}$, whose fluent is $A = My + C$; but when $y = a$, $A = 0$, $\therefore$ $0 = Ma + C$, and $C = -Ma$; consequently $ABDC = My - Ma = AT \times (BD - AC)$. Hence, the whole area $DMB = AT \times BD$, because at an infinite distance $AC = 0$.

Ex. 5. *To find the area of the* catenary *curve* ACB.

Put $CE = x$, $EF = y$, $CF = z$; then $z^2 = 2ax + x^2$ (Prop. 130.), and $z\dot{z} = a\dot{x} + x\dot{x}$; hence, $z^2\dot{z}^2 = (a+x)^2 \times \dot{x}^2$; but $z^2 = 2ax + x^2 = (a+x)^2 - a^2$, and $\dot{x}^2 = \dot{z}^2 - \dot{y}^2$

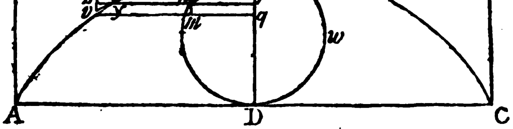
(Prop. 24.); hence, $(a+x)^2 \times \dot{z}^2 - a^2\dot{z}^2 = (a+x)^2 \times (\dot{z}^2 - \dot{y}^2)$, or $a^2\dot{z}^2 = (a+x)^2 \times \dot{y}^2$, and $a\dot{z} = (a+x) \times \dot{y} = a\dot{y} + x\dot{y}$; hence, $x\dot{y} = a\dot{z} - a\dot{y}$; but $flux.\ xy = x\dot{y} + y\dot{x}$; therefore $x\dot{y} = flux.\ xy - y\dot{x}$; hence, $flux.\ xy - y\dot{x} = a\dot{z} - a\dot{y}$, and $\dot{A} = y\dot{x} = flux.\ xy - a\dot{z} + a\dot{y}$; therefore $A = xy - az + ay + C$; but when $x = 0$, then $y = 0$, $z = 0$, and $A = 0$; there-



fore $C = 0$; hence, $A = xy - az + ay = (a+x) \times y - a\sqrt{2ax + x^2}$, the area CEF.

Ex. 6. *To find the area of the* cycloid ABC.

Let BD be the axis, on which describe the circle $BnDw$, draw $rnyz$ perpendicular to BD, and yv a tangent at y; and draw yt, vs perpendicular to FB, and vmq parallel to yr, and mn to qr, and join Bn. Now by the property of the cycloid, the triangles Brn, yzv are similar; hence, Br, or ty, : rn :: zv, or rq, : zy, $\therefore rn \times rq = ty \times zy$, or area $nrqm$ = area $styz$, that is, (Art. 49.) the fluxion of the circular area Bnr = the fluxion of the area Bty; and as these areas begin together at B, and their cotemporary fluxions are

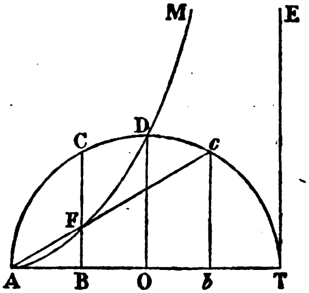


always equal, the quantities generated are equal; hence, the area Bty = the circular area Bnr; bring therefore

yr down to AD, and we have the whole area BFA = the semicircle BnD; hence, $BFA + BEC$ = the whole circle $BnDw$. Now the parallelogram $AFEC = AC \times BD$ = (from the nature of the cycloid) circum. $BnDwB \times BD$ = (by Art. 51. Ex. 3.) four times the area of the whole circle; hence, ABC = three times the whole circle.

Ex. 7. *Let the curve be the* Cissoid *of* Diocles.

On AT describe a semicircle whose center is O, take $AB = bT$, and draw BC, bc, OD, TE, perpendicular to



AT, and join Ac cutting CB in F, and the curve AFM passing through all the points F is the cissoid. Put $AB = x$, $BF = y$, $AT = a$, then $bc^2 = ax - x^2$, and by sim. tri. $x^2 : y^2 :: (a-x)^2 : ax - x^2 :: a - x : x$, and $y^2 = \frac{x^3}{a-x}$; hence, $y\dot{x} = \frac{x^{\frac{3}{2}}\dot{x}}{\sqrt{a-x}}$. Now $bc = \sqrt{ax - x^2}$, and 3 times flux. area $Tbc = 3\dot{x}\sqrt{ax - x^2} = \frac{3ax\dot{x} - 3x^2\dot{x}}{\sqrt{ax - x^2}}$; also, the fluxion of $2\sqrt{ax^3 - x^4}$ is $\frac{3ax\dot{x} - 4x^2\dot{x}}{\sqrt{ax - x^2}}$; therefore 3 times flux. Tbc − flux. $2\sqrt{ax^3 - x^4} = \frac{x^3\dot{x}}{\sqrt{ax - x^2}} = \frac{x^{\frac{3}{2}}\dot{x}}{\sqrt{a-x}} = y\dot{x}$; hence, the area of the part $ABF = 3\,Tbc - 2\sqrt{ax^3 - x^4}$.

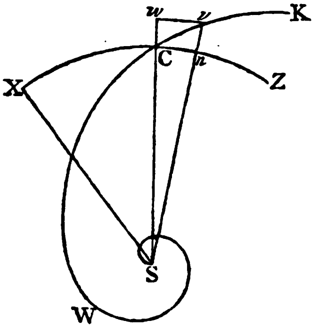
When $x=a$, y is infinite; hence, TE is an asymptote to the curve; and the whole area $=3$ times the area of the semicircle.

To find the AREAS of SPIRALS.

Prop. XXI.

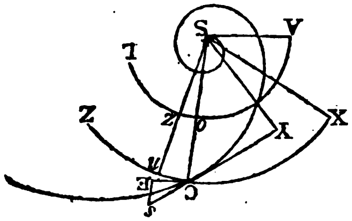
To find the area SWC *of a Spiral.*

(50.) Let $SWCK$ be a spiral, generated by the uniform angular motion of SC about S; SC any ordinate; with the center S describe the circular arc XCZ; draw any other ordinate Sv, and with the center S describe the circular arc vw meeting SC produced in w. Now conceive the sector SXC to have been generated by the uniform angular motion of it's radius about S, at the same time that the area SWC of the spiral was generated by the same uniform angular motion of SC about S. Then SX being constant whilst SC varies, the increment of the sector SXC is the sector SCn, and the cotemporary increment of the area SWC of the spiral is SCv; hence, the ratio of the increment SCn of the sector SXC to the cotemporary increment SCv of the area SWC, is always nearer to a ratio of equality, than



SCn : *Swv*, or nearer than $SC^2 : Sv^2$; * now let *Sv* move up to and coincide with *SC*, in order to obtain the *limiting* ratio of the increments, and we get the *limiting* ratio of $SC^2 : Sv^2$, a ratio of equality; hence, *a fortiori*, the *limiting* ratio of the increment *SCn* to the increment *SCv*, is a ratio of equality; therefore by Prop. 2. the fluxion of the area of the sector *SCX* is equal to the fluxion of the area *SWC* of the spiral; but *SCn* being the increment of the sector *SCX* *uniformly* generated, will represent it's fluxion, by Prop. 1. hence, the fluxion of the area *SWC* of the spiral will be represented by *SCn*.

(51.) Put $SC=y$, the length of the curve $SWC=z$, $XC=x$, $Cn=\dot{x}$, $A=$ the area *SWC*; then the sector $SCn=\frac{y\dot{x}}{2}=\dot{A}$, whose fluent is the area *SWC*. Let *sCY* be a tangent at *C*, and *SY* perpendicular to *CY*; draw *CE* perpendicular to *SC*, and *sE* parallel to *SC*; and with the centre *S* and any radius *SA*, describe a circular arc *AL*. Put $SA=a$, $Ao=w$, $oz=\dot{w}$, $CY=t$, $SY=r$. Then by Art. 31. $Cs=\dot{z}$, $sE=\dot{y}$, $CE=\dot{x}$;



and as the triangles *CEs*, *CSY* are similar, $t : r :: \dot{y} : \dot{x} = \frac{r\dot{y}}{t}$; hence, $SCn=\frac{ry\dot{y}}{2t}=\dot{A}$. Also, by similar sectors *Soz*, *SCn*, $a : y :: \dot{w} : \dot{x} = \frac{y\dot{w}}{a}$; therefore $SCn=\frac{y^2\dot{w}}{2a}=\dot{A}$. These different expressions of the fluxion of the area, are to be used as may be convenient.

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* That similar sectors are as the squares of their radii, appears from Euclid, B. XII. p. 2. and B. VI. p. 33.

EXAMPLES.

Ex. 1. *Let* SWC *be the* logarithmic *spiral; to find it's area.*

Here $r:t$ in a constant ratio, as $m:n$; hence $\dot{A}=\frac{ry\dot{y}}{2t}=\frac{my\dot{y}}{2n}$, whose fluent is $A=\frac{my^2}{4n}+C$; but when $y=0$, $A=0$, $\therefore C=0$; consequently $A=\frac{my^2}{4n}$.

Ex. 2. *Let* SWC *be the spiral of* Archimedes; *to find it's area.*

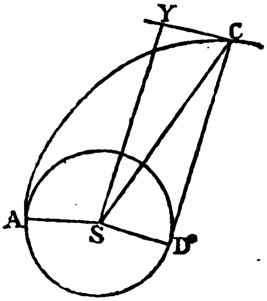
Here $y:w::m:n$, or in a constant ratio; $\therefore$, $\dot{w}=\frac{n\dot{y}}{m}$, consequently $\dot{A}=\frac{y^2\dot{w}}{2a}=\frac{ny^2\dot{y}}{2ma}$, whose fluent is $A=\frac{ny^3}{6ma}+C$; but when $y=0$, $A=0$, $\therefore C=0$; hence, $A=\frac{ny^3}{6ma}$.

Ex. 3. *Let the spiral be a* circle; *to find it's area.*

Here y is constant, and the fluent of $\dot{A}=\frac{y\dot{x}}{2}$ is $A=\frac{yx}{2}$ the area of the sector whose arc is x; hence, if $x=$ the circumference c, the area of the circle $=\frac{cy}{2}$.

Ex. 4. *Let* AC *be the* involute *of the* circle AD, *described by the extremity* C *of a string unwinding itself from the circle; to find it's area.*

It is manifest that DC must be perpendicular to the curve, or to it's tangent CY, and as SD is also perpendicular to CD, and SY to CY, $SDCY$ is a



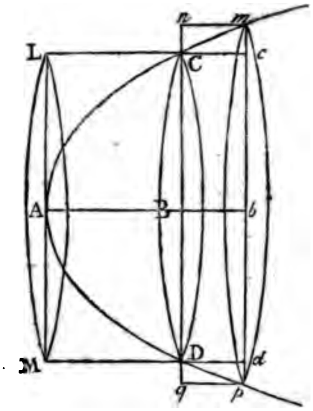
parallelogram, and $SD = CY = t$; hence, $SY = r = \sqrt{y^2 - t^2}$; ∴ $\dot{A} = \frac{ry\dot{y}}{2t} = \frac{\overline{y^2-t^2}|^{\frac{1}{2}} \times y\dot{y}}{2t}$, whose fluent, by Art. 39, is $A = \frac{\overline{y^2-t^2}|^{\frac{3}{2}}}{6t} + C$; but when y (SC) becomes t (SA), then A, or SAC, is $=0$, and $y^2 - t^2 = 0$; hence, $C = 0$; ∴ $SAC = \frac{\overline{y^2-t^2}|^{\frac{3}{2}}}{6t} = \frac{DC^3}{6SD}$.

To find the CONTENTS of SOLIDS.

Prop. XXII.

To find the content of a solid generated by the rotation of a curve about it's axis, or by the motion of a plane parallel to itself.

(52.) Let the solid ACD be conceived to be generated by the uniform motion of the circle CD, beginning at A and increasing in magnitude, having it's plane always perpendicular to AB, and it's centre in that line. Circumscribe this solid by the cylinder $MLCD$, conceived also to be generated at the same time by the same uniform motion of a circle. Then *AL* being constant whilst *BC* varies, let the circle *CD* move on to *mp*, and the solid *CmpD* generated, will be the increment of *ACD*; suppose also the circle *CD* to move on to *cd* in the same time without increasing, and it will generate *CDdc* the cotemporary increment of the cylinder; produce *CD* to *n* and *q*, meeting *mn* and *pq* drawn parallel to *Bb*. Then the ratio of the increment



CDdc of the cylinder to the cotemporary increment *CDpm* of the solid *ACD*, is always nearer to a ratio of equality, than the cylinder *CDdc* : the cylinder *mnqp*, or nearer than $BC^2 : bm^2$. Now let the circle *mp* move up to and coincide with *CD*, in order to obtain the *limiting* ratio of the increments, and we get the *limiting* ratio of $BC^2 : bm^2$, a ratio of equality; hence, *a fortiori*, the *limiting* ratio of the increment *CDdc* of the cylinder, to the cotemporary increment *CDpm* of the solid *ACD*, is a ratio of equality; therefore by Prop. 2. the fluxion of the cylinder *MLCD* is equal to the fluxion of the solid *ACD*; but *CDdc* being the increment of the cylinder *uniformly* generated, will represent it's fluxion, by Prop. 1.; hence, the fluxion of the solid *ACD* will be represented by *CDdc*, the cotemporary fluxion of *AB* being *Bb*. Put therefore $x = AB$, $y = BC$, $\dot{x} = Bb$, $S =$ the solid *ACD*,

$p=3{,}14159$ &c. then (Art. 49. Ex. 2. Cor.) $py^2=$ the area of the circle CBD; hence, the cylinder $CDdc=py^2\dot{x}=\dot{S}$; therefore $S=$ the fluent of $py^2\dot{x}$, corrected if necessary.

The same reasoning will manifestly hold, if the generating plane be any other figure, and continue always parallel to itself. The fluxion therefore of a solid thus generated, will be always expressed by the area of the generating plane and it's velocity conjointly.

EXAMPLES.

Ex. 1. *Let* ACD *be a solid generated by the revolution of any* parabola *about it's axis.*

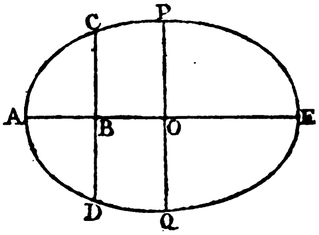
Here $ax=y^n$; hence, $\dot{x}=\frac{ny^{n-1}\dot{y}}{a}$ $\therefore \dot{S}=py^2\dot{x}=\frac{npy^{n+1}\dot{y}}{a}$, whose fluent is $S=\frac{npy^{n+2}}{(n+2)\times a}+C=\frac{n}{n+2}\times py^2\times\frac{y^n}{a}+C=\frac{n}{n+2}\times py^2x+C$; but when $x=0$, $S=0$, $\therefore C=0$; hence, $S=\frac{n}{n+2}\times py^2x$.

If $n=2$, the solid becomes the common paraboloid, and it's content $=\frac{1}{2}py^2x=\frac{1}{2}$ cylinder $LCDM$.

If $n=1$, the curve becomes a straight line, and the solid a cone, and it's content $=\frac{1}{3}py^2x=\frac{1}{3}$ cylinder $LCDM$.

Ex. 2. *Let* APEQ *be a solid generated by the revolution of an* ellipse APEQ *about it's axis* AE.

Put $AB=x$, $BC=y$, $AO=a$, $PO=b$; then by the property of the ellipse, $a^2:b^2::2ax-x^2:y^2=\frac{b^2}{a^2}\times(2ax-x^2)$; hence, $\dot{S}=py^2\dot{x}=\frac{pb^2}{a^2}\times(2ax\dot{x}-x^2\dot{x})$, whose fluent is $S=\frac{pb^2}{a^2}\times(ax^2-\frac{1}{3}x^3)+C$; but when $x=0$,

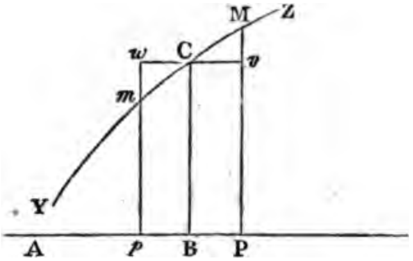


$S=0$, $\therefore C=0$; hence, $S=\frac{pb^2}{a^2}\times(ax^2-\frac{1}{3}x^3)$, which is the solid content of *ACD*; and to get the whole solid, we must make *AB* equal to *AE*, or make $x=2a$; hence, the whole solid $=\frac{pb^2}{a^2}\times(4a^3-\frac{8}{3}a^3)=\frac{4pb^2a}{3}$. If the ellipse revolve about *PQ* instead of *AE*, then, as the same property of the curve holds for each axis, the solid will be $\frac{4pa^2b}{3}$; hence, the solid generated about *AE* : solid about *PQ* $::\frac{4pb^2a}{3}:\frac{4pa^2b}{3}::b:a::$ *PQ* : *AE*.

If $b=a$, the ellipse *APEQ* becomes a circle, and the solid a *sphere*, and the content becomes $=\frac{4pb^3}{3}=4{,}18879b^3$. Now the content of a cylinder circumscribing the sphere = the area of it's end multiplied by it's length = (as the radius of the end $=b$, and length $=2b$) $pb^2\times 2b=2pb^3$; hence, the sphere : cylinder $::\frac{4}{3}:2::2:3$.

Ex. 3. *Let* AP *be the axis of a conic section* YZ *whose general equation is* $y^2=ax^2+bx+c$; *take* BP $=$Bp, *and draw the ordinates* PM, pm; *then the difference of the solids generated by the rotation of the* *area* PMmp *about* Pp, *and the cylinder generated by the parallelogram* PvCmp *about* Pp, *will be a constant quantity*, a *and* x *remaining the same.*

Let $BP=x$, $BC=y$; then $py^2\dot{x}=pax^2\dot{x}+pbx\dot{x}+pc\dot{x}$, and the fluent is $\frac{1}{3}pax^3+\frac{1}{2}pbx^2+cx$=the solid



generated by $BPMC$; and taking x negative $=Bp$, then $y^2=ax^2-bx+c$, and the solid generated by $BCmp$ $=\frac{1}{3}pax^3-\frac{1}{2}pbx^2+cx$; and the sum of these is $\frac{2}{3}pax^3+2pcx$ the solid generated by $PMmp$. Now when $x=0$, $y^2=c=BC^2$; and the cylinder generated by $Pvwp=2py^2x=2pcx$. The difference therefore between the former and the latter is $\frac{2}{3}pax^3$. Wherever therefore BC is taken, this difference remains the same, x and a remaining the same.

For a parabola, $a=0$, and the two solids are always equal.

The frustrum is greater or less than the cylinder, according as a is positive or negative.

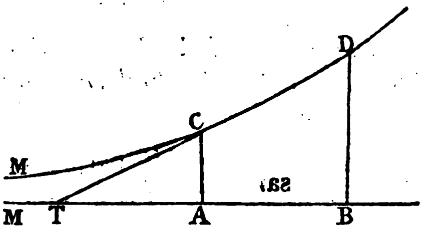
Ex. 4. *To find the content of the solid generated by the revolution of the* Cissoid of Diocles *about it's axis.*

The equation of this curve is $y^2=\dfrac{x^3}{a-x}$, Ex. 7. Prop. XX.); hence, $\dot{S}=py^2\dot{x}=\dfrac{px^3\dot{x}}{a-x}=\dfrac{px^3\dot{x}}{-x+a}=$ (by division) $-px^2\dot{x}-pax\dot{x}-pa^2\dot{x}+\dfrac{pa^3\dot{x}}{a-x}$; now the fluent of all the terms, except the last, is found by Art. 37.

and the fluent of the last, by Art. 45.; hence, the fluent is $S=-\frac{1}{3}px^3-\frac{1}{2}pax^2-pa^2x+pa^3\times-\text{h. l. }(a-x)+C$; now when $x=0$, $S=0$, $\therefore pa^3\times-\text{h. l. }a+C=0$, and $C=pa^3\times\text{h. l. }a$; hence, $S=-\frac{1}{3}px^3-\frac{1}{2}pax^2-pa^2x+pa^3\times-\text{h. l. }(a-x)+pa^3\times\text{h. l. }a=-\frac{1}{3}px^3-\frac{1}{2}pax^2-pa^2x+pa^3\times\text{h. l. }\frac{a}{a-x}$; because $\text{h. l. }a-\text{h. l. }(a-x)=\text{h. l. }\frac{a}{a-x}$, by the nature of Logarithms.

Ex. 5. *To find the content of the solid generated by the* logarithmic *curve* ABDC *revolving about* AB.

Here $y\dot{x}=M\dot{y}$, by Art. 49. Ex. 4. $\therefore \dot{S}=py^2\dot{x}=Mpy\dot{y}$, whose fluent is $S=\frac{Mpy^2}{2}+C$; but when $y=a$, $S=0$,



$\therefore o=\frac{Mpa^2}{2}+C$, and $C=-\frac{Mpa^2}{2}$; hence, $S=\frac{Mp}{2}\times(y^2-a^2)$.

If $y=m$ and n, the area between these two ordinates $=\frac{Mp}{2}\times(m^2-n^2)$, where the correction goes out. Hence, when you want to find the value of a fluent between any two values of the variable quantity, the correction becomes unnecessary.

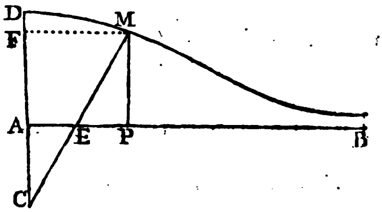
If $AC=a=0$, then $S=\frac{Mpy^2}{2}=$ the whole solid corresponding to the abscissa BM.

Ex. 6. *Let the* catenary *curve revolve about it's axis; to find the content of the solid generated.*

By Prop. 130, $z^2=2ax+x^2$, and therefore $z\dot{z}=a\dot{x}+x\dot{x}$; and by the same Prop. $z\dot{y}=a\dot{x}$. Now $\dot{S}=py^2\dot{x}$; assume therefore $S=py^2x+w$, and we have $\dot{S}=py^2\dot{x}+2pxy\dot{y}+\dot{w}$, and as $\dot{S}=py^2\dot{x}$, we have $\dot{w}=-2pxy\dot{y}=\left(\text{as } \dot{y}=\frac{a\dot{x}}{z}\right)-2pay\times\frac{x\dot{x}}{z}=(\text{as } x\dot{x}=z\dot{z}-a\dot{x})-2pay\times\left(\dot{z}-\frac{a\dot{x}}{z}\right)=-2pay\times(\dot{z}-\dot{y})=2pay\dot{y}-2pay\dot{z}$; assume $w=pay^2-2payz+v$, then $\dot{w}=2pay\dot{y}-2pay\dot{z}-2paz\dot{y}+\dot{v}$; and as $\dot{w}=2pay\dot{y}-2pay\dot{z}$, we have $\dot{v}=2paz\dot{y}=2pa^2\dot{x}$, therefore $v=2pa^2x$; hence, $S=py^2x+pay^2-2payz+2pa^2x\pm C$; but when $x=0$, then $y=0$, $z=0$, and $S=0$, therefore $C=0$; consequently $S=py^2x+pay^2-2payz+2pa^2x$.

Ex. 7. *Let the conchoid* DM *of* Nicomedes *revolve about the axis* DA; *to find the content of the solid generated by* DMF.

Let *CD* be a given line, *AB* perpendicular to it; draw *CEM*, taking $EM=AD$, and the curve



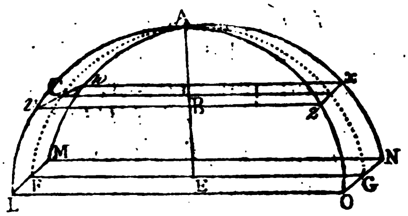
described by *M* is the conchoid. Put $CA=a$, $AD=EM=b$, $AP=x$, $PM=y$; then $CF^2=(a+y)^2$, $EP^2=b^2-y^2$, and by sim. tri. $y^2:b^2-y^2::(a+y)^2:x^2=\frac{(a+y)^2\times(b^2-y^2)}{y^2}$. Now $px^2=$ the area of the circle generated by *FM*, and as $FD=b+y$, $\dot{FD}=-\dot{y}$;

hence, $\dot{S} = -px^2\dot{y} = -p\dot{y} \times \left(\frac{a+y}{y}\right)^2 \times (b^2-y^2) = p \times (a+y)^2 \times \dot{y} - pa^2b^2y^{-2}\dot{y} - pb^2\dot{y} - \frac{2pab^2\dot{y}}{y}$, therefore $S = \frac{p}{3} \times (a+y)^3 + \frac{pa^2b^2}{y} - pb^2y - 2pab^2 \times \text{h. l. } y + C$; now when $y = b$, $S = 0$, and the equation becomes $0 = \frac{p}{3} \times (a+b)^3 + pa^2b - pb^3 - 2pab^2 \times \text{h. l. } b + C$, therefore $C = -\frac{p}{3} \times (a+b)^3 - pa^2b + pb^3 + 2pab^2 \times \text{h. l. } b$; hence, $S = \frac{p}{3} \times (a+y)^3 - \frac{p}{3} \times (a+b)^3 + \frac{pa^2b^2}{y} - pa^2b - pb^2y + pb^3 + 2pab^2 \times \text{h. l. } \frac{b}{y}$ the solid generated by *DMF*.

The solid generated by the whole curve is infinite, as appears by making $y = 0$.

Ex. 8. *Let* LAO *be a solid called a* Groin, *generated by a variable square* v w x z *moving parallel to itself; and let the section* FAG *through the middle of the opposite sides be a semicircle.*

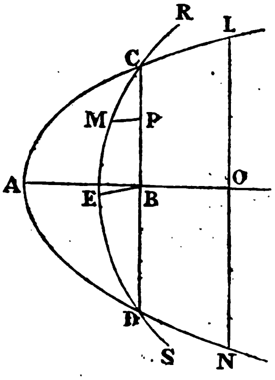
Put $a = AE$, $x = AB$, $y = BC$; then by the property of the circle, $y = \sqrt{2ax - x^2}$, therefore the side of the square $vwxz = 2\sqrt{2ax - x^2}$; hence, the area



$vwxz = 4 \times (2ax - x^2)$, which being the generating plane, it answers to py^2 in the other cases, and therefore $\dot{S} = 4 \times (2ax\dot{x} - x^2\dot{x})$, whose fluent is $S = 4ax^2 - \frac{4}{3}x^3 + C$; but when $x = 0$, $S = 0$, $\therefore C = 0$; hence, $S = 4ax^2 - \frac{4}{3}x^3$, the solid $Avwxz$; and if we make $x = a$, $S = \frac{8a^3}{3}$, the whole solid *ALN*.

If the section *FAG* be any other figure; or if the two sections through the two opposite sides be of different figures, the content may be found in like manner.

Or the solid may be thus generated. Let *LAN* be any curve, *AO* it's axis, *LON*, *CBD* perpendicular to it, *RCEDS* any other curve whose plane is perpendicular to that of *LAN*, and let this curve move from *A* and thus generate a solid. Draw *PM*, *BE*, perpendicular to *DC*, and put $AB = x$, $BC = y$, $CP = v$, $PM = z$; then *$\int z\dot{v}$ = area of *CED* the generating plane; hence, $\int \dot{x} \int z\dot{v}$ = the fluxion of the solid, whose fluent is the solid required.



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* $\int$ This mark denotes *the fluent of.*

Ex. 1. Let LAN be a circle, O it's center, and CED a triangle, and $BE : BC :: m : 1$, and put $a = AO$; then $BC = \sqrt{2ax - x^2}$, and $BE = m\sqrt{2ax - x^2}$; hence, $\int z\dot{v} = m \times (2ax - x^2)$, and the fluxion of the solid $= m \times (2ax\dot{x} - x^2\dot{x})$, whose fluent is $m \times (ax^2 - \frac{1}{3}x^3) =$ solid $ACED$; and when $x = a$, the whole solid $= \frac{2}{3}ma^3 =$ base $\times \frac{2}{3}$ altitude.

Ex. 2. Let DAC, DEC be two parabolas, whose vertices are A, E; and let $px = y^2$, and $m \times BE = y^2$; then $BE = \frac{y^2}{m} = \frac{px}{m}$; also, $y = p^{\frac{1}{2}}x^{\frac{1}{2}}$; therefore $\int z\dot{v} = \frac{4}{3}p^{\frac{1}{2}}x^{\frac{1}{2}} \times \frac{px}{m} = \frac{4p^{\frac{3}{2}}}{3m} \times x^{\frac{3}{2}}$; hence, $\int \dot{x} \int z\dot{v} =$ fluent of $\frac{4p^{\frac{3}{2}}}{3m} \times x^{\frac{3}{2}}\dot{x} = \frac{8p^{\frac{3}{2}}}{15m} \times x^{\frac{5}{2}} = \frac{8}{15} \times \frac{px}{m} \times p^{\frac{1}{2}}x^{\frac{1}{2}} \times x = \frac{8}{15} \times \frac{px}{m} \times y \times x = \frac{4}{3} \times BE \times y \times \frac{2}{5}x =$ area $DEC \times \frac{2}{5}$ altitude.

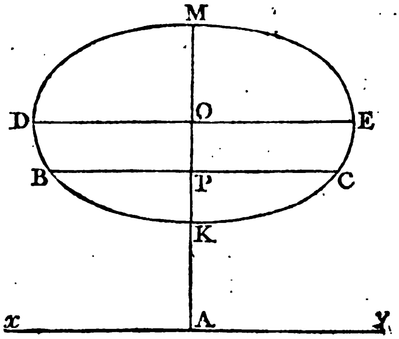
The solid content of bodies may also be found by the following proposition.

Prop. XXIII.

Let DMEK *be any curve revolving about an axis* xy; *then the solid generated, is equal to the circumference described by the centre of gravity multiplied into the area of the figure.*

(53.) Let O be the centre of gravity; draw $MOKA$ perpendicular to xy, and BPC, DOE, parallel to xy. Put $AP = x$, $BC = y$, $AO = a$, $p = 3{,}14159$, &c. Now (Art. 58.) $\frac{\text{flu. } yx\dot{x}}{\text{flu. } y\dot{x}} = a$, $\therefore$ flu. $yx\dot{x} =$ flu. $y\dot{x} \times a =$ area $DKEM \times a$. But the surface generated by $BC = 2pyx$, and therefore the fluxion of the solid $= 2pyx\dot{x}$;

and the solid = $2p \times$ flu. $yx\dot{x} = 2pa \times$ area $DMEK =$



the circumference described by the centre of gravity × area of the figure.

Ex. 1. Let $DMEK$ be a circle, then the solid will represent the ring of an anchor; now in this case, if $r = OM$ the radius, the area $DMEK = pr^2$; hence the solid $= 2p^2ar^2$.

Ex. 2. Let MDK be the given area, and let it be the common parabola, then the center of gravity lies in the axis DO, and therefore $a =$ it's distance from xy; also the area $= \frac{2}{3}DO \times MK$; hence, the solid $= 2pa \times \frac{2}{3}DO \times MK = \frac{4}{3}pa \times DO \times MK$.

Ex. 3. Let MD, DK be straight lines, or MDK a triangle; then the area $= \frac{1}{2}DO \times MK$; hence, the solid $= pa \times DO \times MK$.

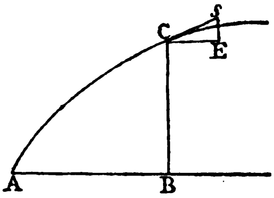
If xy cut $MDKE$, each part must be considered separately.

To find the LENGTHS of CURVES.

Prop. XXIV.

To find the length of a curve line AC, *whose ordinate* BC *is perpendicular to the abscissa* AB.

(54.) Put $AB=x$, $BC=y$, $AC=z$; then if Cs be a tangent to the curve, CE perpendicular to BC, and sE perpendicular to CE, we have, by Art. 27. $CE=\dot{x}$,



$sE=\dot{y}$, $Cs=\dot{z}$; and by Euclid, B. I. p. 47. $\dot{z}^2=\dot{x}^2+\dot{y}^2$, ∴ $\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}$, and $z=$ the fluent of $\sqrt{\dot{x}^2+\dot{y}^2}$, corrected if necessary.

EXAMPLES.

Ex. 1. *Let* AC *be a* semicubical parabola, *whose equation is* $ax^2=y^3$; *to find it's length.*

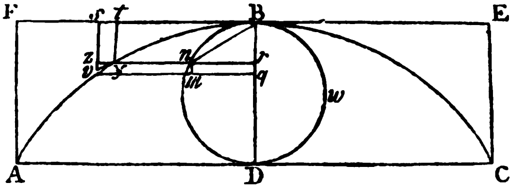
Here $x=\frac{y^{\frac{3}{2}}}{a^{\frac{1}{2}}}$, ∴ $\dot{x}=\frac{3y^{\frac{1}{2}}\dot{y}}{2a^{\frac{1}{2}}}$, ∴ $\dot{x}^2=\frac{9y\dot{y}^2}{4a}$; hence, $\dot{z}^2=\frac{9y\dot{y}^2}{4a}+\dot{y}^2=\left(\frac{9y}{4a}+1\right)\times\dot{y}^2=\frac{9y+4a}{4a}\times\dot{y}^2$, ∴ $\dot{z}=\frac{\overline{9y+4a}\big|^{\frac{1}{2}}\times\dot{y}}{2a^{\frac{1}{2}}}$, whose fluent, by Art. 39. is $z=\frac{\overline{9y+4a}\big|^{\frac{3}{2}}}{27a^{\frac{1}{2}}}+C$; now when $y=0$, $z=0$, in which case, this equation becomes $0=\frac{8a}{27}+C$, ∴ $C=-\frac{8a}{27}$; hence,

$$z=\frac{\overline{9y+4a}\big|^{\frac{3}{2}}}{27a^{\frac{1}{2}}}-\frac{8a}{27}.$$

Ex. 2. *Let* ByA *be a* cycloid; *to find it's length.*

Put $BD=a$, $Br=x$, $By=z$, $yv=\dot{z}$, $vz=rq=\dot{x}$;

then by the prop. of the circle, $Br : Bn :: Bn : BD$, $\therefore$ $Bn^2 = BD \times Br = ax$, and $Bn = a^{\frac{1}{2}}x^{\frac{1}{2}}$; and by the prop.



of the cycloid, x (Br) : $a^{\frac{1}{2}}x^{\frac{1}{2}}$ (Bn) :: $\dot{x}$ (vz) : $\dot{z}$ (vy) $= \frac{a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}}{x} = a^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}$; hence, $z = 2\,a^{\frac{1}{2}}x^{\frac{1}{2}} + C$; but when $x = 0$, $z = 0$, $\therefore$ $C = 0$; consequently $z = 2a^{\frac{1}{2}}x^{\frac{1}{2}} = 2\,Bn$.

Ex. 3. *Let* AC *be the common* parabola; *to find it's length.*

Here $ax = y^2$, $\therefore$ $\dot{x} = \frac{2y\dot{y}}{a} =$ (if $\frac{a}{2} = b$) $\frac{y\dot{y}}{b}$; hence, $\dot{z}^2 = \frac{y^2\dot{y}^2}{b^2} + \dot{y}^2 = \frac{y^2+b^2}{b^2} \times \dot{y}^2$, $\therefore$ $\dot{z} = \frac{\overline{y^2+b^2}|^{\frac{1}{2}} \times \dot{y}}{b}$ = (by multiplying numerator and denominator by $y \times \overline{y^2+b^2}|$) $\frac{1}{b} \times \frac{y^3\dot{y}+b^2y\dot{y}}{\overline{y^4+b^2y^2}|^{\frac{1}{2}}} = \frac{1}{2b} \times \frac{2y^3\dot{y}+2b^2y\dot{y}}{\overline{y^4+b^2y^2}|^{\frac{1}{2}}} = \frac{1}{2b} \times \frac{2y^3\dot{y}+b^2y\dot{y}}{\overline{y^4+b^2y^2}|^{\frac{1}{2}}}$ $+ \frac{1}{2b} \times \frac{b^2y\dot{y}}{\overline{y^4+b^2y^2}|^{\frac{1}{2}}} =$ (by dividing the num. and den. of the last term by y) $\frac{1}{2b} \times \overline{y^4+b^2y^2}|^{-\frac{1}{2}} \times (2y^3\dot{y}+b^2y\dot{y}) + \frac{1}{2}b$ $\times \frac{\dot{y}}{\overline{y^2+b^2}|^{\frac{1}{2}}}$; now the fluent of the first term is $\frac{1}{2b} \times$ $\overline{y^4+b^2y^2}|^{\frac{1}{2}}$, by Art. 39. and the fluent of the last term is $\frac{1}{2}b\times$ h.l.$(y+\overline{y^2+b^2}\big|^{\frac{1}{2}})$, by Art. 45. Ex. 4. hence, $z=\frac{1}{2b}\times\overline{y^4+b^2y^2}\big|^{\frac{1}{2}}+\frac{1}{2}b\times$ h. l. $(y+\overline{y^2+b^2}\big|^{\frac{1}{2}})+C$; now when $y=0$, $z=0$, in which case, the equation is $0=\frac{1}{2}b\times$ h. l. $b+C$; hence, $C=-\frac{1}{2}b$ h. l. b; therefore $z=\frac{1}{2b}\times\overline{y^4+b^2y^2}\big|^{\frac{1}{2}}+\frac{1}{2}b\times$ h. l.$\frac{y+\overline{y^2+b^2}\big|^{\frac{1}{2}}}{b}$.

Ex. 4. *To find the length* CD *of any part of the* logarithmic curve. (See Fig. Prop. XXII. Ex. 4.)

Put $AC=a$, $AB=x$, $BD=y$, $CD=z$; then $\frac{M\dot{y}}{y}=\dot{x}$ (Art. 49. Ex. 4.); $\therefore\ \dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}=\sqrt{\frac{M^2\dot{y}^2}{y^2}+\dot{y}^2}=\frac{\dot{y}\sqrt{M^2+y^2}}{y}=$ (by multiplying the numerator and denominator by $\sqrt{M^2+y^2}$) $\frac{\dot{y}\times(M^2+y^2)}{y\sqrt{M^2+y^2}}=\frac{y\dot{y}}{\sqrt{M^2+y^2}}+\frac{M^2\dot{y}}{y\sqrt{M^2+y^2}}=\frac{y\dot{y}}{\sqrt{M^2+y^2}}+\frac{M^2y^{-2}\dot{y}}{\sqrt{1+M^2y^{-2}}}$; hence, (by Prop. 16. and Prop. 18. Ex. 8.) $z=\sqrt{M^2+y^2}-M\times$ h. l. $\frac{M+\sqrt{M^2+y^2}}{My}+C$; but when $z=0$, $y=b$, and we have $0=\sqrt{M^2+b^2}-M\times$ h. l. $\frac{M+\sqrt{M^2+b^2}}{Mb}+C$; hence, $C=-\sqrt{M^2+b^2}+M\times$ h. l. $\frac{M+\sqrt{M^2+b^2}}{Mb}$; therefore $z=\sqrt{M^2+y^2}-\sqrt{M^2+b^2}+M\times$ h. l. $\frac{M+\sqrt{M^2+b^2}}{Mb}-M\times$ h. l.

$$\frac{M+\sqrt{M^2+y^2}}{My} = \sqrt{M^2+y^2} - \sqrt{M^2+b^2} + M \times \text{h. l.}\ \frac{My+y\sqrt{M^2+b^2}}{Mb+b\sqrt{M^2+y^2}}.$$

Ex. 5. *To find the length of a* circular *arc*.

By Art. 46. $\dot{z} = \frac{a^2\dot{t}}{a^2+t^2}$ = (by division) $\dot{t} - \frac{t^2\dot{t}}{a^2} + \frac{t^4\dot{t}}{a^4} - \frac{t^6\dot{t}}{a^6}$ + &c. hence, $z = t - \frac{t^3}{3a^2} + \frac{t^5}{5a^4} - \frac{t^7}{7a^6}$ + &c. $+C$; but when $t=0$, $z=0$, therefore $C=0$; hence, $z = t - \frac{t^3}{3a^2} + \frac{t^5}{5a^4} - \frac{t^7}{7a^6}$ + &c. Now if $a=1$, and z be an arc of 30°, then $t = \sqrt{\tfrac{1}{2}} = 0{,}5773502$, which being substituted for t, if we take 12 terms of this series, we get $z = 0{,}5235987$, the length of an arc of 30°; which multiplied by 12 gives 6,2831804 for the length of the circumference of a circle whose radius is unity.

If we take the arc $z = 45°$, then will $t = a$; hence, $z = a \times (1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{\&c.})$

Ex. 6. *To find the length of an elliptic arc.*

Put (fig. to Prop. 10. Ex. 2.) $AO=a$, $OP=b$, $OB=x$, $BC=y$, then $y = \frac{b}{a}\sqrt{a^2-x^2}$, $\dot{y} = \frac{-bx\dot{x}}{a\sqrt{a^2-x^2}}$; hence, $\dot{z} = \sqrt{\dot{x}^2+\dot{y}^2} = \frac{\dot{x}\sqrt{a^4-a^2x^2+b^2x^2}}{a\sqrt{a^2-x^2}}$ = (if $d = \frac{a^2-b^2}{a^2}$) $\dot{x}\sqrt{\frac{a^2-dx^2}{a^2-x^2}}$ = (by division and extracting the square root) $\dot{x} + \frac{1-d}{2a^2}x^2\dot{x} + \frac{3-2d-d^2}{8a^4}x^4\dot{x}$ + &c. and

$z = x + \frac{1-d}{6a^2}x^3 + \frac{3-2d-d^2}{40a^4}x^5 + \&c. =$ the arc PC.

Ex. 7. *To find the length of an hyperbolic arc.*

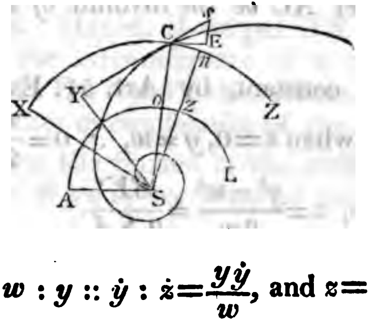
Put (fig. to Prop. 10. Ex. 3.) $AO = a$, semi-minor axis $= b$, $x = OB$, $y = BC$, then $y = \frac{b}{a}\sqrt{x^2 - a^2}$, and $x = \sqrt{b^2 + \frac{b^2}{a^2}y^2}$, $\dot{x} = \frac{by\dot{y}}{\sqrt{a^4 + a^2y^2}}$; hence, $\dot{z} = \dot{y}\sqrt{\frac{a^2+dy^2}{a^2+y^2}}$ (putting $d = \frac{a^2+b^2}{a^2}$) $= \dot{y} - \frac{1-d}{2a^2}y^2\dot{y} + \frac{3-2d-d^2}{8a^4}y^4\dot{y} -$ &c. and $z = y - \frac{1-d}{6a^2}y^3 + \frac{3-2d-d^2}{40a^4}y^5 -$ &c. $=$ arc AC.

To find the LENGTHS of SPIRALS.

Prop. XXV.

To find the length of a spiral SC.

(55.) Let the ordinate $SC = y$, the curve $SC = z$, $CY = w$; then, by Art. 31. $CS = \dot{z}$, $Es = \dot{y}$; and by



sim. triangles, $w : y :: \dot{y} : \dot{z} = \frac{y\dot{y}}{w}$, and $z =$ the fluent of $\frac{y\dot{y}}{w}$, corrected if necessary.

EXAMPLES.

Ex. 1. *Let* SC *be the* logarithmic *spiral; to find it's length.*

Here $w : y :: m : n$, a constant ratio; hence $w = \frac{my}{n}$, $\therefore \dot{z} = \frac{n\dot{y}}{m}$, and $z = \frac{ny}{m} + C$; but when $y=0$, $z=0$, $\therefore$ $C=0$; consequently $z = \frac{ny}{m} = \frac{y^2}{w}$; therefore $CY : CS :: CS :$ the length of the curve.

Ex. 2. *Let it be the spiral of* Archimedes; *to find it's length.*

By Art. 32. Ex. 1. $w = \frac{ty}{\sqrt{y^2+t^2}}$; hence, $\dot{z} = \frac{\dot{y}\sqrt{y^2+t^2}}{t}$, which is the same as the fluxion of the length of the parabolic arc, Art. 54. Ex. 3. $\therefore$ $z = \frac{1}{2t} \times \overline{y^4+t^2y^2}|^{\frac{1}{2}} + \frac{1}{2}t \times \text{h. l.} \frac{y+\sqrt{y^2+t^2}}{t}$.

Ex. 3. *Let* AC *be the* involute *of a* circle; *to find it's length.*

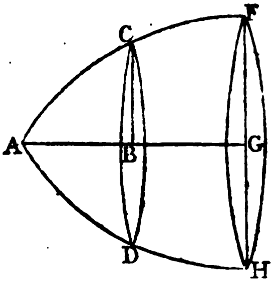
Here w is constant, by Art. 51. Ex. 4. hence, $z = \frac{y^2}{2w} + C$; but when $z=0$, $y=w$, $\therefore$ $0 = \frac{w^2}{2w} + C$, and $C = -\frac{w^2}{2w}$; hence, $z = \frac{y^2-w^2}{2w} = \frac{SY^2}{2SA}$.

To FIND THE SURFACES OF SOLIDS.

PROP. XXVI.

To find the surface of a solid generated by the rotation of a curve about it's axis, or by the motion of a plane parallel to itself.

(56.) Conceiving the solid *AFH* to be generated as in Art. 52. by the circle *CD*, the surface may be considered as generated by the periphery of that circle; the fluxion therefore of the surface will be the periphery of the circle multiplied by the velocity with which it flows, by Cor. Art. 49. But the velocity with which any point *C* of the periphery flows, is the velocity with which *AC* increases at the point *C*, or it is $\dot{z}$, putting $AC=z$. Hence, if we put $AB=x$, $BC=y$, $p=6{,}28318$ &c. the circumference of a circle whose radius $=1$ (Art. 54. Ex. 5.), $S=$ the surface *ACD*; then $1 : y :: p : py$ the circumference of the circle *CD*; therefore $\dot{S} = py\dot{z}$ the fluxion of the surface; consequently the fluent of $py\dot{z}$, corrected if necessary, will be the surface.



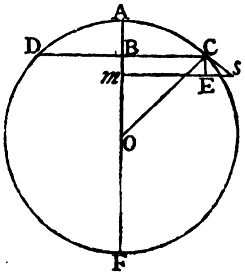
The method of finding the fluxion of the surface of a solid, may be further illustrated thus.

Let *ACF* be protended into a straight line, and let an ordinate perpendicular to it, and always equal to the periphery of the circle *CD*, move from *A* to *F* with the same velocity as the point C, upon the solid, moves; then it is manifest, that the area generated by this ordinate must always be equal to the area generated by the periphery of the circle, the generating lines and their velocities being always equal, and both moving in directions perpendicular to themselves; hence, the fluxion of the surface $ACD =$ the fluxion of the area of this curve $=$ (by Art. 49.) the ordinate multiplied by the fluxion of the abscissa $=$ the periphery of the circle CD multiplied by the fluxion of the curve AC.

EXAMPLES.

Ex. 1. *Let* ADFC *be a sphere whose centre is* O; *to find it's surface.*

Let Cs be a tangent at C, sEm parallel to BC, and



CE to Bm; then if $AB = x$, $BC = y$, $AC = z$, by Art. 23. $Cs = \dot{z}$, $CE = \dot{x}$; and by similar triangles CEs, CBO, $\dot{z} : \dot{x} :: a : y$, $\therefore y\dot{z} = a\dot{x}$; hence, $\dot{S} = py\dot{z} = pa\dot{x}$, the fluxion of the surface DAC, whose fluent $S = pax + C$; but when $x = 0$, $S = 0$, $\therefore$ $C = 0$; hence, $S = pax$ the surface DAC. If we make AB equal to AE, or $x = 2a$, we have $2pa^2$ for the whole surface of the sphere. Now if we conceive $ADFC$ to be a great circle of the sphere, it's area $= \frac{1}{2}pa^2$, by Art. 49. Ex. 2. Cor. Hence, the whole surface of a sphere is equal to four times the area of a great circle of that sphere.

Cor. As the surface $DAC=pax$, it varies as x.

Ex. 2. *Let the solid* AFH *be generated by the common* parabola; *to find it's surface.*

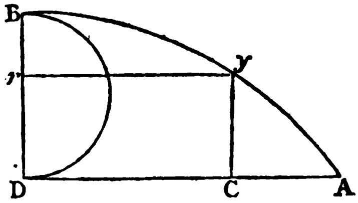
Here $ax=y^2$; hence, $\dot{x} = \frac{2y\dot{y}}{a}$, and $\dot{x}^2=\frac{4y^2\dot{y}^2}{a^2}$; ∴ (Prop. 24.) $\dot{z}^2=\dot{x}^2+\dot{y}^2=\frac{4y^2\dot{y}^2}{a^2}+\dot{y}^2=\left(\frac{4y^2}{a^2}+1\right)\times\dot{y}^2 = \frac{4y^2+a^2}{a^2}\times\dot{y}^2$, and $\dot{z}=\frac{\overline{4y^2+a^2}|^{\frac{1}{2}}\times\dot{y}}{a}$; hence, $\dot{S}=py\dot{z}=\frac{p\times\overline{4y^2\times a^2}|^{\frac{1}{2}}\times y\dot{y}}{a}$, whose fluent, by Art. 39. is $S=\frac{p\times\overline{4y^2+a^2}|^{\frac{3}{2}}}{12a}+C$; now when $y=0$, $S=0$, in which case, the equation becomes $0=\frac{pa^3}{12}+C$; hence, $C=-\frac{pa^3}{12}$; therefore $S=\frac{p\times\overline{4y^2+a^2}|^{\frac{3}{2}}}{12a}-\frac{pa^2}{12}$.

Ex. 3 *Let* ALN *be a* groin, *as in Art.* 52. *Ex.* 8. *to find it's surface.*

Put $AB=x$, $BC=y$, $AC=z$; and we have (Art. 46.) $\dot{z}=\frac{a\dot{x}}{\sqrt{2ax-x^2}}$; also, $vw = 2BC, = 2\sqrt{2ax-x^2}$; now vw is the line generating one of the four surfaces; hence, $8\sqrt{2ax-x^2}$ answers to py in the other cases; therefore if S be the surface Avx, $\dot{S} = 8a\dot{x}$, and $S=8ax+C$; but when $x=0$, $S=0$, ∴ $C=0$; consequently $S=8ax$; and when $x=a$, $S=8a^2$.

Ex. 4. *To find the surface generated by the revolution of the* cycloidal curve BA *about its base* DA.

Put $By=z$, $Br=x$, $rD=Cy=y$, $BD=a$; then, by Art. 54. Ex. 2. $\dot{z}=a^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}$; $\therefore \dot{S}=py\dot{z}=pya^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}=p\times(a-x)\times a^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}=pa^{\frac{3}{2}}x^{-\frac{1}{2}}\dot{x}-pa^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}$; hence, $S=$



$2pa^{\frac{3}{2}}x-\frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}}+C$; but when $x=0$, $S=0$, $\therefore C=0$; hence, $S=2pa^{\frac{3}{2}}x^{\frac{1}{2}}-\frac{2}{3}pa^{\frac{1}{2}}x^{\frac{3}{2}}$ the surface generated by By; and when $x=a$, we have $S=\frac{4pa^2}{3}$, the whole surface generated by BA.

Ex. 5. *To find the surface of the solid generated by any part* CD *of the* logarithmic *curve revolving about it's axis* AB.

By Prop. 24. Ex. 4. $\dot{z}=\frac{\dot{y}\sqrt{M^2+y^2}}{y}$, therefore $\dot{S}=py\dot{z}=p\dot{y}\sqrt{M^2+y^2}$, which fluxion is the same as that for the value of $\dot{z}$ in Prop. 24. Ex. 3. (the constant multiplier and divisor excepted); therefore $S=\frac{p}{2}\times\sqrt{y^4+M^2y^2}+\frac{pM^2}{2}\times \text{h. l. }(y+\sqrt{M^2+y^2})+C$; but when $y=a$, $S=0$; hence, $0=\frac{p}{2}\times\sqrt{a^4+M^2a^2}+\frac{pM^2}{2}\times \text{h. l. }(a+\sqrt{M^2+a^2})+C$, and $C=-\frac{p}{2}\times\sqrt{a^4+M^2a^2}-\frac{pM^2}{2}\times\text{h.l.}(a+\sqrt{M^2+a^2})$;

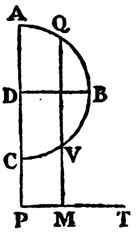
therefore $S = \frac{p}{2} \times \sqrt{y^4 + M^2y^2} - \frac{p}{2} \times \sqrt{a^4 + M^2a^2}$ $+ \frac{pM^2}{2} \times \text{h. l.} \frac{y + \sqrt{M^2 + y^2}}{a + \sqrt{M^2 + a^2}}$.

Ex. 6. *To find the surface of the solid generated by the* catenary *curve revolving about it's axis.*

By Prop. 130. we have $z^2 = 2ax + x^2$; hence, $a^2 + 2ax + x^2 = a^2 + z^2$, and $a + x = \sqrt{a^2 + z^2}$; therefore $\dot{x} = \frac{z\dot{z}}{\sqrt{a^2 + z^2}}$, and $\dot{y} = \sqrt{\dot{z}^2 - \dot{x}^2} = \frac{a\dot{z}}{\sqrt{a^2 + z^2}}$. Now $\dot{S} = py\dot{z}$; assume $S = pyz - w$, then $\dot{S} = py\dot{z} + pz\dot{y} - \dot{w}$, and as $\dot{S} = py\dot{z}$, we have $\dot{w} = pz\dot{y} = \frac{paz\dot{z}}{\sqrt{a^2 + z^2}}$, whose fluent is $w = pa\sqrt{a^2 + z^2}$ (Prop. 16.); hence, $S = pyz - pa\sqrt{a^2 + z^2} + C = pyz - pa^2 - pax + C$, but when $x = 0$, $y = 0$, and $S = 0$, therefore $C - pa^2 = 0$, and $C = pa^2$; hence, $S = pyz - pax$ the surface generated by the curve CF revolving about the axis CE.

Let ABC *be any curve,* DB *it's axis perpendicular to* AC, *and* PT *perpendicular to* AC *produced; to find the surface of the solid generated by* ABC *revolving bout* PT.

Put $PD = a$, draw $M\dot{q}D$ parallel to AP, and let



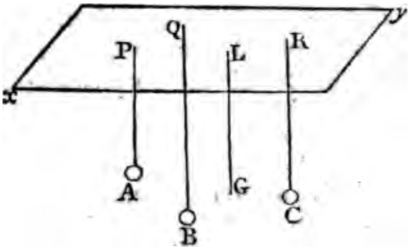
$BQ = Bq = z$; then $\dot{S} = p \times MQ \times \dot{z} + p \times Mq \times \dot{z} = 2pa\dot{z}$, and the fluent is $S = 2paz$; and for the whole surface, $S = pa \times ABC$.

If ABC be a circle whose radius $= r$, then it's circumference $= pr$; hence, $S = p^2ar$ the surface of a circular ring, whose thickness is $2r$, and inward radius $a - r$.

SECT. V.

ON THE CENTRE OF GRAVITY.

(57.) If there be any number of bodies *A*, *B*, *C*, and *G*

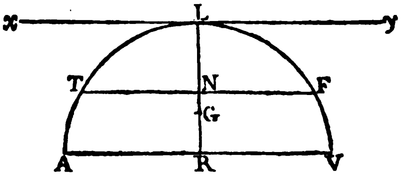


be their centre of gravity; and to any plane *xy*, perpendiculars *AP*, *BQ*, *CR*, *GL* be let fall, then (*Mechanics*, Art. 173.) $LG = \frac{A \times AP + B \times BQ + C \times CR}{A + B + C}$.

PROP. XXVII.

To find the centre of gravity of a body, considered as an area, solid, surface of a solid, or curve line.

(58.) Let *ALV* be any curve, *RL* the axis in which the centre of gravity must lie; for as it bisects every ordinate *TF* in *N*, the parts on each side *LR* will always balance each other, and therefore the body will balance itself upon *LR*; consequently the centre of gravity must be somewhere in that line. Put $LN = x$, $TN = y$, $TL = z$, and draw *xy* parallel to *TF*; then if we conceive this body to be made up of an indefinite number of corpuscles, and multiply



each corpuscle by it's perpendicular distance from xy, the sum of all the products divided by the sum of all the corpuscles, or by the whole body, will give LG by Art. 57. Now to get the sum of all these products, we must first get the fluxion of the sum, and the fluent will be the sum itself. Put $\dot{s}$ for the fluxion of the body at the distance x from xy, then will $x\dot{s}$ be the fluxion of the sum of all the products; also, $\dot{s}$ is the fluxion of the sum of all the corpuscles; therefore by Art. 57. $LG=\frac{\text{flu. } x\dot{s}}{\text{flu. } \dot{s}}$.

1st. If the body be an *area*, then $\dot{s}=2y\dot{x}$ by Art. 49; hence, $LG=\frac{\text{flu. } 2yx\dot{x}}{\text{flu. } 2y\dot{x}}=\frac{\text{flu. } yx\dot{x}}{\text{flu. } y\dot{x}}$.

2nd. If the body be a *solid*, then $py^2\dot{x}=\dot{s}$ by Art. 52; hence, $LG=\frac{\text{flu. } py^2x\dot{x}}{\text{flu. } py^2\dot{x}}=\frac{\text{flu. } y^2x\dot{x}}{\text{flu. } y^2\dot{x}}$.

3rd. If the body be the *surface of a solid*, then $\dot{s}=py\dot{z}$ by Art. 56; hence, $LG=\frac{\text{flu. } pyx\dot{z}}{\text{flu. } py\dot{z}}=\frac{\text{flu. } yx\dot{z}}{\text{flu. } y\dot{z}}$.

4th. If the body be a *curve line FT*, then $\dot{s}=2\dot{z}$; hence, $LG=\frac{\text{flu. } 2x\dot{z}}{\text{flu. } 2\dot{z}}=\frac{\text{flu. } x\dot{z}}{\text{flu. } \dot{z}}=\frac{\text{flu. } x\dot{z}}{z}$.

EXAMPLES.

Ex. 1. *Let* $y=ax^n$ *be the equation to any* parabola; *to find it's centre of gravity.*

As $y=ax^n$, $\therefore yx\dot{x}=ax^{n+1}\dot{x}$, whose fluent is $\frac{ax^{n+2}}{n+2}$;

also, $y\dot{x} = ax^n\dot{x}$, whose fluent is $\frac{ax^{n+1}}{n+1}$; hence, (Art. 58.) $LG = \frac{ax^{n+2}}{n+2} \times \frac{n+1}{ax^{n+1}} = \frac{n+1}{n+2} \times x$.

If $n = \frac{1}{2}$, then $y = ax^{\frac{1}{2}}$, $\therefore y^2 = a^2x$, which is the common parabola; hence, $LG = \frac{3}{5}x$.

If $n = 1$, then $y = ax$, and the figure is a triangle; hence, $LG = \frac{2}{3}x$.

Ex. 2. *Let* $y = ax^n$; *to find the centre of gravity of the solid generated by the revolution of this curve about it's axis.*

As $y^2 = a^2x^{2n}$, $\therefore y^2x\dot{x} = a^2x^{2n+1}\dot{x}$, whose fluent is $\frac{a^2x^{2n+2}}{2n+2}$; also, $y^2\dot{x} = a^2x^{2n}\dot{x}$, whose fluent is $\frac{a^2x^{2n+1}}{2n+1}$; hence, by Article 58, $LG = \frac{a^2x^{2n+2}}{2n+2} \times \frac{2n+1}{a^2x^{2n+1}} = \frac{2n+1}{2n+2} \times x$.

If $n = \frac{1}{2}$, the solid becomes a paraboloid, and $LG = \frac{2}{3}x$.

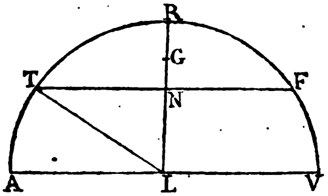
If $n = 1$, the solid becomes a cone, and $LG = \frac{3}{4}x$.

Ex. 3. *Let* ALV *be a* hemispheroid; *to find it's centre of gravity.*

Put $LR = a$, $AR = b$; then $a^2 : b^2 :: 2ax - x^2 : y^2 = \frac{b^2}{a^2} \times (2ax - x^2)$; hence, $y^2x\dot{x} = \frac{b^2}{a^2} \times (2ax^2\dot{x} - x^3\dot{x})$, whose fluent is $\frac{b^2}{a^2} \times (\frac{2}{3}ax^3 - \frac{1}{4}x^4)$; also, $y^2\dot{x} = \frac{b^2}{a^2} \times (2ax\dot{x} - x^2\dot{x})$, whose fluent is $\frac{b^2}{a^2} \times (ax^2 - \frac{1}{3}x^3)$; hence, by Art. 58. $LG = \frac{\frac{2}{3}ax^3 - \frac{1}{4}x^4}{ax^2 - \frac{1}{3}x^3}$; and when $x = a$, $LG = \frac{\frac{2}{3}a^4 - \frac{1}{4}a^4}{a^3 - \frac{1}{3}a^3} = \frac{5a}{8}$ for the whole solid. As this is independent of b, if $b = a$, *LG* remains the same, and the solid becomes an *hemisphere.*

Ex. 4. *Let* ARV *be a* semicircle; *to find it's centre of gravity.*

Put $LN=x$, $TN=y$, $TL=r$; then $x^2+y^2=r^2$; hence,



$x\dot{x}+y\dot{y}=0$, $\therefore yx\dot{x}=-y^2\dot{y}$, whose fluent is $-\frac{1}{3}y^3+C$, which must vanish when *TF* coincides with *AV*, or $y=r$; therefore put r for y, and $-\frac{1}{3}r^3+C=0$, $\therefore C=\frac{1}{3}r^3$; hence, the correct fluent of $yx\dot{x}$ is $\frac{1}{3}r^3-\frac{1}{3}y^3$; also, the fluent of $y\dot{x}$ is (Art. 49.) the area *ATNL*; hence, by Art. 58. $LG=\frac{1}{3}\times\frac{r^3-y^3}{ATNL}$; and when $y=0$, $LG=\frac{r^3}{3ARL}$ for the semicircle.

Ex. 5. *Let* ARV *be the* periphery *of a semicircle; to find it's center of gravity.*

Put $LN=x$, $NT=y$, $AT=z$, then, (Art. 46.) $\dot{z}:-\dot{y}::r:x$, and $x\dot{z}=-r\dot{y}$, whose fluent is $-ry+C$, which must vanish when $y=r$, or $-r^2+C=0$, and $C=r^2$; hence, the correct fluent is r^2-ry; therefore (Art. 58.) $LG=\frac{r^2-ry}{z}$, and when $y=0$, $LG=\frac{r^2}{RA}$.

Ex. 6. *To find the centre of gravity of the* surface ARV *of an* hemisphere.

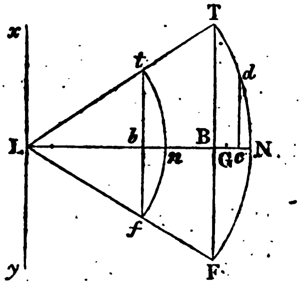
Put $x=RN$, $y=TN$, $z=RT$, and $a=TL$; then

(Art. 46.) we have $\dot{z} : \dot{x} :: a : y$, therefore $y\dot{z} = a\dot{x}$; hence, $yx\dot{z} = ax\dot{x}$, whose fluent is $\frac{1}{2}ax^2$; also, the fluent of $y\dot{z}$, or $a\dot{x}$, is ax; hence, by Art. 58. $RG = \frac{\frac{1}{2}ax^2}{ax} = \frac{1}{2}x$; and when $x = RL = r$, then $RG = \frac{1}{2}r$ for the hemisphere.

If the line in which the centre of gravity lies, do not bisect the figure, or the parts on each side be not similar, for instance, if we take the figure *LAR*; then, as before, calculate the distance of the centre of gravity from *LR*, in a line parallel to *LR*; and from *LA* in a line parallel to *LA*; and the intersection of these lines will give the centre of gravity of *ALR*.

Ex. 7. *To find the centre of gravity of the* circular arc TNF.

Let *L* be the centre, and *LN* bisect the arc *TF* in *N*, join *TF* which bisects *LN* in *B*, and draw *dc* perpendicular to *LN*. Put $LT = r$, $TB = m$, arc $TN = c$, $Lc = x$, $cd = y$, $dN = s$; then (Art. 46.) $\dot{s} : \dot{y} :: r : x$, and $x\dot{s} = r\dot{y}$, whose fluent is ry, and when $y = m$ it becomes rm, and $2rm$ for the whole arc *TNF*; hence, $LG = \frac{2rm}{2c} = \frac{rm}{c}$.



Ex. 8. *To find the centre of gravity of the* Sector LTF.

Describe the circular arc *tnf*, and draw *tbf*. Put

$Lt = x$, arc $tn = z$, then $r : x :: m : \frac{mx}{r} = tb$; and by the last Art. the sum of the products of each particle of tnf multiplied into it's perpendicular dist. from xy (to which LN is perpendicular) is $2\,Lt \times tb = \frac{2mx^2}{r}$; hence for the sector, the fluxion of that product is $\frac{2mx^2\dot{x}}{r}$, whose fluent is $\frac{2mx^3}{3r}$; which, divided by $LTNF = xz$, gives $\frac{2mx^2}{3rz} =$ (when $x = r$) $\frac{2mr}{3z} = LG$.

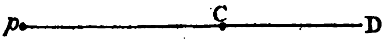
◆

On the CENTRE of GYRATION.

DEFINITION.

(59.) The centre of gyration is that point of a body revolving about an axis, into which if the whole quantity of matter were collected, the same moving force would generate the same angular velocity in the body.

(60.) Let a corpuscle p revolve about C, and let a force act at D to oppose its motion. Then the momentum of p varies as $p \times$ it's velocity, or as $p \times pC$, which we may consider as a power acting at p in opposition to



the force at D; but this power acting at the distance pC from the centre of motion, it's effect to oppose a force at D must (by the property of the lever) be as $p \times pC \times pC = p \times pC^2$. This effect of p to persevere in it's motion, or which is the same, to prevent any change in it's motion, is called it's *inertia*.

PROP. XXVIII.

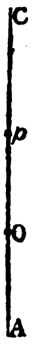
To find the centre of gyration *of a body.*

(61.) Let a body be conceived to be made up of the particles A, B, C, &c. whose distances from the axis are a, b, c, &c. and let x be the distance of the centre of gyration from the axis, then by Art. 59. the inertia of A, B, C, &c. will be as $A\times a^2$, $B\times b^2$, $C\times c^2$, &c. and the inertia of all the matter at the distance x will be as $(A+B+C+\text{\&c.})\times x^2$; now as the moving force is the same in both cases, the inertia must be the same when the same angular velocity is generated; hence, $(A+B+C+\text{\&c.})\times x^2=A\times a^2+B\times b^2+C\times c^2+\text{\&c.}$ therefore $x=\sqrt{\dfrac{A\times a^2+B\times b^2+C\times c^2+\text{\&c.}}{A+B+C+\text{\&c.}}}$; that is, if $\dot{s}$ be the fluxion of the body at the distance z from the axis $x=\sqrt{\dfrac{\text{flu. } z^2\dot{s}}{s}}$.

EXAMPLES.

Ex. 1. *Let the* straight line CA *revolve about* C; *to find* O *the centre of gyration.*

Put $z=Cp$, then $s=z$, and $\dot{s}=\dot{z}$, $\therefore z^2\dot{s}=z^2\dot{z}$, whose

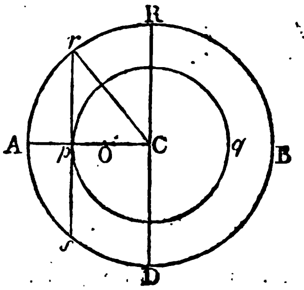


fluent is $\frac{1}{3}z^3 =$ (when $z = CA$) $\frac{1}{3}CA^3$; hence, $CO = \sqrt{\frac{1}{3}CA^2} = CA\sqrt{\frac{1}{3}}$.

If it were required to find the inertia of any part pA, if $Cp = a$, the fluent $\frac{1}{3}z^3$ would want a correction to make it vanish when $z = a$; the inertia therefore of $pA = \frac{1}{3} \times (z^3 - a^3) = \frac{1}{3} \times (CA^3 - Cp^3)$; hence, $CO = \sqrt{\frac{1}{3} \times \frac{CA^3 - Cp^3}{Ap}}$.

Ex. 2. *Let a* circle AB *revolve in it's own plane about it's centre* C; *to find* O *it's centre of gyration.*

Put $p = 6{,}28318$, &c. the circumference of a circle whose radius $= 1$, $z = Cp$; then the circumference $pq = pz$, and $pz\dot{z} = \dot{s}$; hence, the fluent of $pz\dot{z} \times z^2$, or



$pz^3\dot{z}$, is $\frac{1}{4}pz^4 =$ (when $z = CA = r$) $\frac{1}{4}pr^4$. Also, the area of the circle $= \frac{1}{2}pr^2$; hence, $CO = \sqrt{\frac{1}{2}r^2} = r\sqrt{\frac{1}{2}}$.

Cor. The same must be true for a *cylinder* revolving about it's axis, it being true for every section parallel to the end.

Ex. 3. *Let* RADB *be a* sphere *revolving about the diameter* RD; *to find* O *it's centre of gyration.*

Draw CA perpendicular and spr parallel to RD; put $Cr = r$, $Cp = z$, then $pr = \sqrt{r^2 - z^2}$; and if $p = 6{,}28318$,

e. the surface of the cylinder generated by sr revolving bout RD, is $pz \times 2\sqrt{r^2 - z^2}$; hence, $\dot{s} = 2pz\dot{z}\sqrt{r^2 - z^2}$, nd $z^2\dot{s} = 2pz^3\dot{z}\sqrt{r^2 - z^2}$. Now to find this fluent, put $-z^2 = y^2$, then $z^2 = r^2 - y^2$, and $z^4 = r^4 - 2r^2y^2 + y^4$, $\therefore z^3\dot{z} = -r^2y\dot{y} + y^3\dot{y}$; hence, $2pz^3\dot{z}\sqrt{r^2 - z^2} = 2p \times$ $(-r^2y^2\dot{y} + y^4\dot{y})$, whose fluent is $2p \times (-\frac{1}{3}r^2y^3 + \frac{1}{5}y^5)$, and hen $z = 0$ this fluent ought to vanish, but y is then $= r$, and the fluent becomes $2p \times -\frac{2}{15}r^5$; hence, the correct fluent is $2p \times \left(\frac{2}{15}r^5 - \frac{1}{3}r^2y^3 + \frac{1}{5}y^5\right)$; and the whole fluent when $z = r$ (in which case $y = 0$) will be $\frac{4}{15}pr^5$. ow the content of the sphere $= \frac{2}{3}pr^3$; hence, $CO =$ $\sqrt{\frac{2}{5}r^2} = r\sqrt{\frac{2}{5}}$.

◆

ON THE CENTRE OF PERCUSSION.

DEFINITION.

(62.) The centre of percussion, is that point in the is * of a revolving body, which striking against an mmoveable obstacle, the whole motion, estimated in he *plane* of the body's motion, shall be destroyed.

PROP. XXIX.

To find the centre of percussion *of a body.*

(63.) Let ABD be a plane passing through the ntre of gravity G of the body, and perpendicular to e axis of suspension which passes through C; and nceive the whole body to be projected upon this plane lines perpendicular to it, or parallel to the axis;

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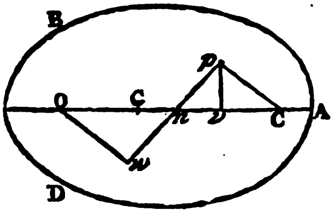
[*] The axis is here understood to be a right line drawn through e centre of gravity of the body, perpendicular to the axis about hich the body revolves.

then as each particle is thus kept at the same distance from the axis, the effect, from the rotatory motion about the axis, will not be altered, nor will the centre of gravity be changed. Let O be the centre of percussion,



and draw pnw perpendicular to pC, and Ow perpendicular to pw; also, pv perpendicular to Cn. As the velocity of any particle $p \propto pC$, the momentum of p in the direction $pw \propto p \times pC$, it being as the velocity and quantity of matter conjointly; and by the property of the lever, the efficacy of this force to turn the body about O is as $p \times pC \times Ow =$ (because $On : Ow :: pC : vC$) $p \times vC \times On = p \times vC \times (CO - Cn) = p \times vC \times CO - p \times vC \times Cn =$ (as $Cn : Cp :: Cp : vC$) $p \times vC \times CO - p \times Cp^2$. Now that the efficacy of all the particles to turn the body about O may be $= 0$, we must make the *sum of all the quantities* $p \times vC \times CO$ − *sum of all the quantities* $p \times Cp^2 = 0$; hence, $CO =$

$$\frac{\textit{sum of all the } p \times Cp^2}{\textit{sum of all the } p \times vC} = \frac{\textit{sum of all the } p \times Cp^2}{\textit{body} \times CG},$$

these two denominators being equal from the property of the centre of gravity (Art. 57.). By the next Prop. it appears that this centre is the same as the centre of oscillation; the calculations therefore made for that centre, apply to this centre.

Although the body by striking at O may have no tendency to move in the *plane* of it's previous motion, and this only is included in the common definition which we here follow, yet it may have a tendency to revolve about AO. If therefore we were to define the centre of percussion to be that point where the *whole* motion would be destroyed, we must find the plane parallel to *ABD*, such that the sum of all the forces to turn the body about the line joining the centre of percussion and the axis of vibration in that plane, is also $= 0$. But this is a problem not fit for an elementary treatise.—See the *Hydrostatics*, fourth edit. Prob. To find the Centre of Pressure.

(64.) As the force acting at *O* destroys the motion, let us suppose a force to act at *O* and to generate the motion back again; then it is manifest, that the body would *begin* to return under all the same circumstances in which it's motion ceased; that is, it would *begin* it's motion by revolving about *C*. In this case, *C* is called the centre of *spontaneous* rotation; making therefore the point at which a force acts upon a body that can move freely, the centre of percussion, the centre of spontaneous rotation coincides with the centre of rotation corresponding to that centre of percussion.

On the CENTRE of OSCILLATION.

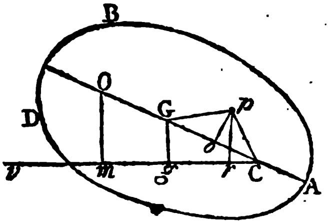
DEFINITION.

The centre of oscillation, is that point in the axis of a vibrating body, at which if a particle were suspended from the axis of motion, it would vibrate in the same time the body does.

Prop. XXX.

To find the centre of oscillation *of a body.*

(65.) Let *ABD* be a body projected upon a plane perpendicular to the axis of rotation, as in Art. 63. the axis passing through *C* and supposed to be parallel to the horizon; and let *G* be the centre of gravity, *O* the centre of oscillation; draw *Cv* parallel to the horizon, *Om*, *Gg*, *pr* perpendicular to it. Then by the property of the lever, the force of gravity to turn the particle p about $C \propto p \times Cr$; hence, the force of gravity to turn the whole body about $C \propto$ the *sum of all*



the $p \times Cr$. Also, the force of gravity to turn a single particle O at O about $C \propto O \times Cm$. Now by Art. 60. the inertia of $p \propto p \times Cp^2$, and therefore the inertia of the whole body $\propto$ the *sum of all the* $p \times Cp^2$. Also, the inertia of $O \propto O \times OC^2$. Now that the acceleration of the body about C may be equal to that of the particle O, the moving forces must be in proportion to the inertiæ; because, if the powers to produce motion be as the powers to oppose it, the acceleration must be the same. Hence, *sum of all* $p \times Cr : O \times Cm :: \textit{sum of all } p \times Cp^2 : O \times OC^2$, therefore $OC = \dfrac{\textit{sum of all } p \times Cp^2 \times Cm}{\textit{sum of all } p \times Cr \times OC} = \dfrac{\textit{sum of all } p \times Cp^2}{\textit{body} \times CG}$, because (by sim. triangles) $Cm : CO :: Cg : CG$, and therefore $\dfrac{Cm}{CO} = \dfrac{Cg}{CG}$, and by the property of the centre of gravity, *sum of all* $p \times Cr = \textit{body} \times Cg$. Hence, the centre of oscillation is the same as the centre of percussion. Or if s be the body, x the distance of $\dot{s}$ from the axis of suspension, then $CO = \dfrac{\text{flu. } x^2 \dot{s}}{\text{flu. } x \dot{s}} = \dfrac{\text{flu. } x^2 \dot{s}}{s \times CG}$. The point O is therefore a fixed point for every part of the vibration.

(66.) Join pG; and draw Po perpendicular to CG;

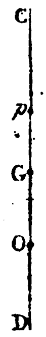
then $Cp^2 = CG^2 + Gp^2 - 2\,CG \times Go$, $\therefore p \times Cp^2 = p \times CG^2 + p \times Gp^2 - 2CG \times p \times Go$, and the *sum of all* $\text{p} \times \text{Cp}^2 =$ *sum of all* $\text{p} \times \text{CG}^2 +$ *sum of all* $\text{p} \times \text{Gp}^2 - 2\text{CG} \times$ *sum of all* $\text{p} \times \text{Go}$; but the *sum of all* $\text{p} \times \text{Go} = 0$, from the property of the centre of gravity; and the *sum of all* $\text{p} \times \text{CG}^2 = \textit{body} \times \text{CG}^2$; hence, *sum of all* $\text{p} \times \text{Cp}^2 = \textit{body} \times \text{CG}^2 +$ *sum of all* $\text{p} \times \text{Gp}^2$; consequently $CO = \dfrac{\textit{body} \times \text{CG}^2 + \textit{sum of all}\ \text{p} \times \text{Gp}^2}{\textit{body} \times \text{CG}} = \text{CG} + \dfrac{\textit{sum of all}\ \text{p} \times \text{Gp}^2}{\textit{body} \times \text{CG}}$; hence, $GO = \dfrac{\textit{sum of all}\ \text{p} \times \text{Gp}^2}{\textit{body} \times \text{CG}}$. Now as the numerator is constant, GO varies inversely as CG; hence, if we find GO for any one value of CG, we shall know every other value of GO from that of CG. Hence also, if O be the centre of suspension, C will become the centre of oscillation; for as $GO \times GC$ is constant, if C be changed to O, O must be changed to C.

Cor. If x be the distance from C to the centre of gyration; then by Art. 61. $x^2 s =$ *sum of all* $\text{p} \times \text{Cp}^2$; and by Art. 65. $CO \times s \times CG =$ *sum of all* $\text{p} \times \text{Cp}^2$; hence, $x^2 = CO \times CG$, and $CG : x :: x : CO$.

EXAMPLES.

Ex. 1. *Let* CD *be a* straight line *suspended at* C; *to find the centre* O *of oscillation.*

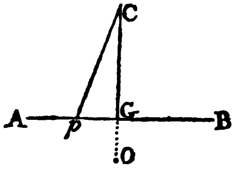
Put $x = Cp$; then the fluent of $x^2\dot{s} = \text{flu.}\ x^2\dot{x} = \frac{1}{3}x^3$



$= (\text{when } x = CD)\ \frac{1}{3}CD^3$. Also, *body* $\times CG = CD \times \frac{1}{2}CD = \frac{1}{2}CD^2$; hence, $CO = \frac{2}{3}CD$.

Ex. 2. *Let the* line AB *vibrate* lengthways *in a vertical plane about* C, *which is equidistant from* A *and* B; *to find it's centre* O *of oscillation.*

Draw CG perpendicular to AB; and put $CG = a$, $Gp = x$; then $pC^2 = a^2 + x^2$; and the fluent of $Cp^2 \times \dot{s}$ $=$ fluent of $a^2\dot{x} + x^2\dot{x} = a^2x + \frac{1}{3}x^3 = (\text{when } x = AG)\ a^2 \times AG + \frac{1}{3}AG^3$; hence, for the whole line AB, it becomes $2a^2 \times AG + \frac{2}{3}AG^3$. Also, *body* $\times CG = a \times AB = a \times 2AG$; hence, $CO = \frac{2a^2 \times AG + \frac{2}{3}AG^3}{a \times 2AG} = a + \frac{AG^2}{3a}$.

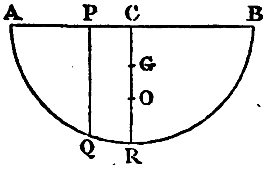


Ex. 3. *Conceive* ARB (see the figure to the next Example) *an angular rod vibrating about an axis* AB *parallel to the horizon; to find the length of the pendulum.*

Draw PQ, CR perpendicular to AB; put $AR = m$, $BR = n$, $b = CR$, $x = AQ$; then $m : x :: b : PQ = \frac{bx}{m}$; and the fluent of $\frac{b^2x^2\dot{x}}{m^2}$ is $\frac{b^2x^3}{3m^2}$, which, when $x = m$, becomes $\frac{b^2m}{3}$ expressing the sum of the products of each particle of AR, multiplied into the square of their distances from the axis. In like manner $\frac{b^2n}{3}$ is the sum for BR. Also, $(m+n)\times\frac{b}{2}$ is the whole body multiplied into the distance of the centre of gravity from the axis. Hence, CO the length of the pendulum $=$ $\frac{\frac{1}{3}b^2\times(m+n)}{\frac{1}{2}b\times(m+n)}=\frac{2}{3}b=\frac{2}{3}CR$. The given rod therefore vibrates in the same time CR would.

Ex. 4. *Let a rod* ARB *in the form of the* semicircumference of a circle, *revolve about* AB *parallel to the horizon*; *to find the length of the pendulum.*

Let C be the centre, draw PQ, CR perpendicular to AC, and let G be the centre of gravity of the rod.

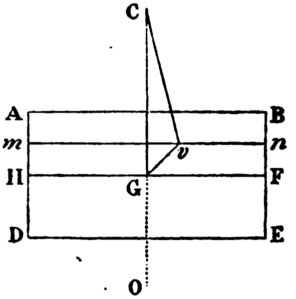


Put $r=CA$, $x=PQ$, $z=AQ$; then (Art. 46.) $\dot{z}=\frac{r\dot{x}}{\sqrt{r^2-x^2}}$; hence, $x^2\dot{z}=\frac{rx^2\dot{x}}{\sqrt{r^2-x^2}}$, whose fluent (Prop. 63.) is $\frac{1}{2}r^2z-\frac{1}{2}r\sqrt{r^2x^2-x^4}$; and when $x=r$, it becomes $\frac{1}{2}r^2\times AR$; and for ARB the fluent becomes $r^2\times AR$. Also (Art. 58. Ex. 5.) $CG=\frac{r^2}{AR}$. Hence (Art. 65.), the length CO of the pendulum $=r^2\times AR\div\frac{r^2}{AR}\times 2AR=\frac{1}{2}AR$.

Ex. 5. *Let* ABED *be a parallelogram,* CG *perpendicular to the plane,* G *the centre,* EGF *parallel to*

AB, *and the axis of vibration parallel to* HF; *to find* CO *the length of the pendulum.*

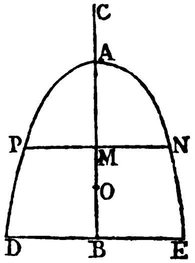
Let $AB = b$, $BE = 2n$, $CG = a$, draw mn parallel to HF, and Gv perpendicular to it, and put $Gv = x$; then



$Cv^2 = a^2 + x^2$; and for $mnFH$, we have the fluxion of the sum of all the particles × the square of their distances $= b \times (a^2\dot{x} + x^2\dot{x})$, whose fluent (when $x = n$) is $ba^2n + \frac{1}{3}bn^3$; and for $ABED$ it is $2ba^2n + \frac{2}{3}bn^3$. Hence, $CO = \dfrac{2ba^2n + \frac{2}{3}bn^3}{2bna} = a + \frac{1}{3}\dfrac{n^2}{a}$.

Ex. 6. *Let* DAE *be any* parabola *vibrating* flatways, *or about an axis passing through* C *parallel to* PMN; *to find the centre* O *of oscillation.*

Put $AC = d$, $AM = x$, $PM = y$, then $ax^n = y$; hence, $2y\dot{x} = 2ax^n\dot{x} = \dot{s}$; and the fluent of $CM^2 \times \dot{s}$, or



$2.(d+x)^2 \times ax^n\dot{x}$, or $2d^2ax^n\dot{x} + 4dax^{n+1}\dot{x} + 2ax^{n+2}\dot{x}$, is $\dfrac{2d^2ax^{n+1}}{n+1} + \dfrac{4dax^{n+2}}{n+2} + \dfrac{2ax^{n+3}}{n+3}$, which vanishes when

$x = 0$, and therefore it wants no correction. Also, the fluent of $CM \times \dot{s}$, or $(d+x) \times 2ax^n\dot{x}$ is $\frac{2dax^{n+1}}{n+1} + \frac{2ax^{n+2}}{n+2}$; hence, if the former be divided by the latter, we get (by reduction) $CO =$

$$\frac{(n+2).(n+3).d^2 + (n+1).(n+3).2dx + (n+1).(n+2).x^2}{(n+2).(n+3).d + (n+1).(n+3).x}.$$

If $d = 0$, and $n=1$, the figure becomes a triangle, and $AO = \frac{3}{4}x$.

If $n = \frac{1}{2}$, it becomes the common parabola, and $AO = \frac{5}{7}x$.

If AB were a straight rod, and the density were as AM^n, the conclusion would be the same.

Ex. 7. *Let the* parabola *vibrate* edgeways, *and let it be suspended at* A ; *to find the centre of oscillation.*

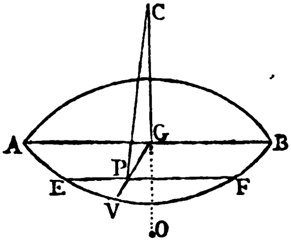
By Ex. 2. the sum of the products of each particle of the line PN into the square of it's distance from A, is $2x^2 \times y + \frac{2}{3}y^3 = 2x^2 \times ax^n + \frac{2}{3}a^3x^{3n}$; hence, $2ax^{n+2}\dot{x} + \frac{2}{3}a^3x^{3n}\dot{x}$ is the fluxion of the sum of the products for the whole body; whose fluent is $\frac{2ax^{n+3}}{n+3} + \frac{2a^3x^{3n+1}}{3.3(n+1)}$. Also, the fluent of $AM \times \dot{s}$ is the same as before, d being now$=0$; hence, $AO = \frac{(n+2).x}{n+3} + \frac{a^2.(n+2).x^{2n-1}}{3.(3n+1)}$.

If $n = \frac{1}{2}$, it is the common parabola, and $AO = \frac{5x}{7} + \frac{a^2}{3}$.

If $n=1$, $AO = \frac{3x}{4} + \frac{a^2x}{4}$ for a triangle; and if $a = 1$, $AO = x$.

Ex. 8. *Let* CG *be perpendicular to the plane of the circle* ABV, *and let the* circle *vibrate about an axis passing through* C *and parallel to* AB; *to find the centre* O *of oscillation.*

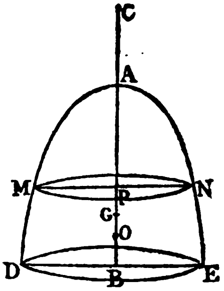
Draw GPV perpendicular to AB, and EF parallel to AB. Put $AG=r$, $CG=a$, $GP=x$, then $CP^2=a^2+x^2$, $PE=\sqrt{r^2-x^2}$, and $EF=2\sqrt{r^2-x^2}$; hence,



$EF\times CP^2=(a^2+x^2)\times 2\sqrt{r^2-x^2}$, which multiplied by $\dot{x}$ gives $(a^2\dot{x}+x^2\dot{x})\times 2\sqrt{r^2-x^2}$ for the fluxion of the sum of the products of each particle of the area $ABFE$ multiplied into the square of it's distance from the axis of vibration. Now to find the fluent, we have the fluent of $a^2\times 2\sqrt{r^2-x^2}\times\dot{x}=a^2\times$ area $ABFE$ by Art. 49. and when $x=r$, the fluent $=a^2\times AVB$; and as the same is true for the other semicircle, the whole fluent is $a^2\times$ circle AEB. The fluent of the second part, $2x^2\dot{x}\sqrt{r^2-x^2}$, may be found thus. Let $\dot{x}\sqrt{r^2-x^2}=\dot{A}$, $x^2\dot{x}\sqrt{r^2-x^2},=\dot{B}$, and $x\times\overline{r^2-x^2}\big|^{\frac{3}{2}}=P$; then by taking the fluxion of the last, we have $\dot{P}=\dot{x}\times\overline{r^2-x^2}\big|^{\frac{3}{2}}-3x^2\dot{x}\sqrt{r^2-x^2}=\dot{x}\times(r^2-x^2)\times\sqrt{r^2-x^2}-3x^2\dot{x}\sqrt{r^2-x^2}=r^2\dot{x}\sqrt{r^2-x^2}-4x^2\dot{x}\sqrt{r^2-x^2}$, that is, $\dot{P}=r^2\dot{A}-4\dot{B}$, hence, (by taking the fluents) $P=r^2A-4B$, and $B=\frac{r^2A-P}{4}$; therefore the fluent of $2x^2\dot{x}\sqrt{r^2-x^2}$ is $\frac{r^2A-P}{2}$; but when $x=r$, $P=0$; and the fluent becomes $\frac{r^2A}{2}=\frac{r^2}{8}\times$ circle AEB, because $A=\frac{1}{4}$ of the circle when $x=r$; and for both semicircles it becomes $\frac{r^2}{4}\times$ circle; hence, the whole fluent is $(a^2+\frac{1}{4}r^2)\times$ circle, which is the sum of the products of each particle of the circle × the square of it's distance from the axis of vibration. Also, $a\times$ circle $=$ the denominator for the value of CO; hence, by dividing the former by the latter, we get $CO=a+\frac{r^2}{4a}$.

Ex. 9. *Let the solid formed by the rotation of any curve* DAE *about it's axis* AB, *vibrate about* C *in* BA *produced; to find the centre* O *of oscillation.*

By Ex. 8. the sum of the products of each particle of the circle MN into the square of it's distance from the axis $=(CP^2+\frac{1}{4}PN^2)\times$ circle $MN=(CP^2+\frac{1}{4}PN^2)\times p\times PN^2$($p$ being $=3.14159$ &c.) $=p\times(CP^2\times PN^2+\frac{1}{4}PN^4)=p\times(\overline{d+x}|^2\times y^2+\frac{1}{4}y^4)$; hence, $p\dot{x}\times$



$((d+x)^2\times y^2+\frac{1}{4}y^4)$ is the fluxion of the sum of all such products for the whole body; the fluent of which divided by $CG\times$ *body*, gives CO.

Ex. 10. *Let the solid be a* paraboloid; *to find the centre of oscillation.*

Here $ax=y^2$; hence, $p\dot{x}((d+x)^2\times y^2+\frac{1}{4}y^4)$ is equal to

$p\dot{x} \times \left((d+x)^2 \times ax + \frac{1}{4}a^2x^2\right)$, whose fluent is $\frac{1}{2}pad^2x^2 + \frac{2}{3}padx^3 + \frac{1}{4}pax^4 + \frac{1}{12}pa^2x^3$; also (Art. 53. Ex. 1.), the body $= \frac{1}{2}pax^2$; and (Art. 58. Ex. 2.) $AG = \frac{2}{3}x$; $\therefore$ $CG = d + \frac{2}{3}x$; hence, $CG \times body = \frac{1}{2}padx^2 + \frac{1}{3}pax^3$; dividing therefore the above fluent by this quantity, we have $CO = \dfrac{6d^2 + 8dx + 3x^2 + ax}{6d + 4x}$.

If C coincide with A, $d = 0$, and $CO = \dfrac{3x + a}{4}$.

Ex. 11. *Let the solid be a* cone; *to find the centre of oscillation.*

Put $AB = a$, $BD = b$; then $a : b :: x : y = \dfrac{bx}{a} = \left(\text{if } m = \dfrac{b}{a}\right) mx$; hence, $p\dot{x} \times \left((d+x)^2 \times y^2 + \frac{1}{4}y^4\right) = p\dot{x} \times \left((d+x)^2 \times m^2x^2 + \frac{1}{4}m^4x^4\right)$, whose fluent is $\frac{1}{3}pd^2m^2x^3 + \frac{1}{2}pdm^2x^4 + \frac{1}{5}pm^2x^5 + \frac{1}{20}pm^4x^5$; also (Art. 53. Ex. 1.), the body $= \frac{1}{3}pm^2x^3$; and (Art. 58. Ex. 2.) $AG = \frac{3}{4}x$, $\therefore CG = d + \frac{3}{4}x$; hence, $CO = \dfrac{20d^2 + 30dx + 12x^2 + 3m^2x^2}{20d + 15x} = \dfrac{20d^2 + 30da + 12a^2 + 3b^2}{20d + 15a}$ for the whole cone, when $x = a$, and $mx = y = b$.

If the cone be suspended at the vertex, then $d = 0$, and $CO = \dfrac{4a^2 + b^2}{5a}$.

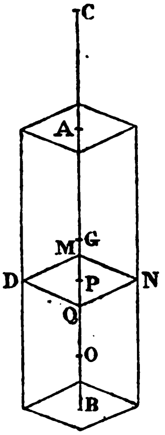
Ex. 12. *Let the solid be a* cylinder *whose axis is* AB.

Put $PN = r$ a constant quantity, $CA = a$, $CB = b$, $CP = x$; then $(CP^2 + \frac{1}{4}PN^2) \times \text{circle } MN = \left(x^2 + \dfrac{r^2}{4}\right) \times pr^2$; hence, the fluxion of the sum of all the particles $\times$ the square of their distance from the axis $= \left(x^2\dot{x} + \dfrac{r^2\dot{x}}{4}\right) \times pr^2$, whose fluent $= \left(\dfrac{x^3}{3} + \dfrac{r^2x}{4}\right) \times pr^2$; and this fluent for the length of the cylinder, or between the values of $x = a$ and $x = b$, is $\left(\frac{b^3 - a^3}{3} + \frac{r^2}{4} \times (b - a)\right) \times p r^2$; also, $s \times CG = pr^2 \times (b - a) \times \frac{1}{2}(b + a)$; and dividing the former by the latter, we get $CO = \frac{4b^2 + 4ab + 4a^2 + 3r^2}{6b + 6a}$.

If $r = 0$, $CO = \frac{2b^2 + 2ab + 2a^2}{3b + 3a}$ for a rod of an evanescent diameter. If $a = 0$, $CO = \frac{2}{3} b$.

Ex. 13. *Let the solid be a* parallelopipedon *whose axis is* AB.

Let the axis of vibration be parallel to MN one side of the section; put $MN = m$, $DM = 2n$, $x = CP$,



$a = CA$, $b = CB$; then by Ex. 5. $2mnx^2\dot{x} + \frac{2}{3}mn^3\dot{x}$ is the fluxion of the sum of all the particles $\times d$ into the squares of their distances from the axis, and the fluent is $\frac{2}{3}mnx^3 + \frac{2}{3}mn^3x$; and as in the last example, for the whole solid, this becomes $\frac{2}{3}mn \times (b^3 - a^3) + \frac{2}{3}mn^3 \times (b - a)$; and $s \times CG = 2mn \times (b - a) \times \frac{1}{2}(b + a)$; hence,

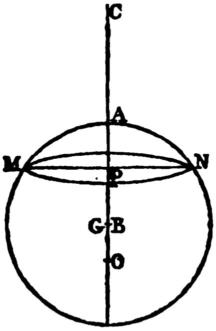
$$CO = \frac{\frac{2}{3}mn \times (b^3 - a^3) + \frac{2}{3}mn \times (b - a)}{2mn \times (b - a) \times \frac{1}{2}(b + a)} = \frac{2}{3} \times \frac{b^2 + ab + a^2 + n^2}{b + a}.$$

If $n = 0$, $a = 0$, $CO = \frac{2}{3}b$.

This being independent of m, it is just the same as it would be for a rectangle whose breadth is $2n$, vibrating edgeways.

Ex. 14. *Let the body be a* sphere ; *to find the centre* O *of oscillation,* C *being the point of suspension.*

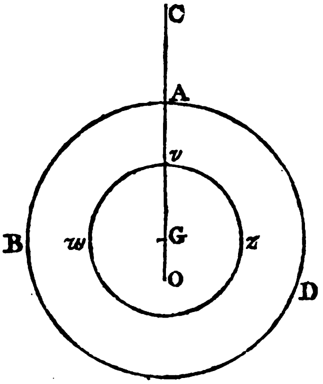
Let B be the centre ; then $BA = r$, $y^2 = 2rx - x^2$.



In this case, it will be most convenient to apply the rule in Art. 66. that is, to get the value of CO when C coincides with A, and thence to deduce it's value in any other case. Now when C coincides with A, $d = 0$, and the expression becomes $p\dot{x} \times (x^2y^2 + \frac{1}{4}y^4) = p \times (r^2x^2\dot{x} + rx^3\dot{x} - \frac{3}{4}x^4\dot{x})$, whose fluent is $\frac{1}{3}pr^2x^3 + \frac{1}{4}prx^4 - \frac{3}{20}px^5$; and when $x = 2r$ it becomes $\frac{28}{15}pr^5$ for the whole sphere. Also, the *body* × CG (G coinciding with B) $= \frac{4}{3}pr^3 \times r = \frac{4}{3}pr^4$; therefore $AO = 1\frac{2}{5}r$; consequently $BO = \frac{2}{5}r$. Hence, (Art. 66.) if $d = CB$, $d : r :: \frac{2}{5}r : \frac{2r^2}{5d} = BO$ when the point of suspension is at C; therefore $CO = d + \frac{2r^2}{5d}$.

Ex. 15. *Let the body be a* circle, *and the axis of vibration pass through* C *perpendicular to it's plane.*

Put $GA=r$, $CG=d$, $GO=x$, and $p=6{,}283$ &c. then



px = the circumference vwz, and the fluxion of the sum of all the particles multiplied into the square of their distances from $G = px \times x^2 \times \dot{x}$, whose fluent, when $x=r$, is $\frac{pr^4}{4}$; and the area of the circle $\times\, d = \frac{pr^2}{2} \times d$; hence, (Art. 66.) $CO = d + \frac{r^2}{2d}$.

If C coincide with A, then $CO = \frac{3}{2}r$.

Cor. Hence, the same must be true for a *cylinder* whose axis is parallel to the axis of vibration.

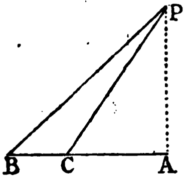
SECT. VI.

ON THE ATTRACTIONS OF BODIES.

PROP. XXXI.

To determine the attraction of a corpuscle P *towards a* right line BA, *in the direction* PA *perpendicular to* AB, *supposing the attraction to each particle of the line to vary inversely as the square of the distance.*

(67.) PUT $PA=a$, $AC=x$, then $PC^2=a^2+x^2$, and therefore the attraction of P towards a particle at C is as $\frac{1}{a^2+x^2}$; and by the resolution of forces $\sqrt{a^2+x^2} : a$



$:: \frac{1}{a^2+x^2} : \frac{a}{\overline{a^2+x^2}|^{\frac{3}{2}}}$ the attraction in the direction PA;

hence, $\frac{a\dot{x}}{\overline{a^2+x^2}|^{\frac{3}{2}}}$ is the fluxion of the whole force, whose fluent (Art. 39. Ex. 5.) is $\frac{x}{\overline{a^2+x^2}|^{\frac{1}{2}} \times a}$, which wants no correction, for when $x=0$, the fluent $=0$; and when $x=AB$, it becomes $\frac{AB}{PB \times PA}$ for the whole attraction in the direction PA.

If AB be infinite, the attraction is as $\frac{1}{PA}$.

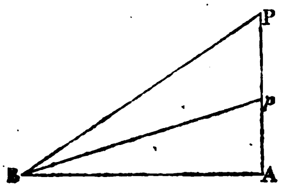
In like manner we find the whole attraction in the direction AB; for $\sqrt{a^2+x^2} : x :: \frac{1}{a^2+x^2} : \frac{x}{\overline{a^2+x^2}|^{\frac{3}{2}}}$, and the fluxion of the force is $\frac{x\dot{x}}{\overline{a^2+x^2}|^{\frac{3}{2}}}$, whose fluent (Art. 39.) is $-\frac{1}{\overline{a^2+x^2}|^{\frac{1}{2}}}$, which wants a correction, for when $x=0$, it becomes $-\frac{1}{a}$; hence, the correct fluent is $\frac{1}{a}-\frac{1}{\overline{a^2+x^2}|^{\frac{1}{2}}}$, and when $x=AB$, it becomes $\frac{1}{PA}-\frac{1}{PB}=\frac{PB-PA}{PB\times PA}$ for the whole attraction in the direction AB.

Hence, the attraction in the direction PA : the attraction in the direction AB :: $AB : PB-PA$; take therefore $AC=PB-PA$, and join PC, and that will be the direction in which the corpuscle P will *begin* to move.

PROP. XXXII.

If the line PA *be perpendicular to the line* BA; *to find the attraction of* PA *to* BA, *upon the same law of force.*

(68.) Put $a=AB$, $x=Ap$; then (Art. 67.) the attraction



of a corpuscle at p to $AB=\frac{a}{x\sqrt{a^2+x^2}}$; hence, $\frac{a\dot{x}}{x\sqrt{a^2+x^2}}$ is the fluxion of the attraction required; whose fluent

(Art. 45. Ex. 7.) is $\frac{1}{2}$ h. l. $\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}$; now when $x=0$, this becomes $\frac{1}{2}$ h. l. $\frac{a-a}{a+a}=\frac{1}{2}$ h. l. $\frac{o}{2a}$; hence, the fluent corrected, or the attraction, is $\frac{1}{2}$ h. l. $\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x}+a}$ − $\frac{1}{2}$ h. l. $\frac{o}{2a}$ = (when $x = AP$) $\frac{1}{2}$ h. l. $\frac{BP-AB}{AB+BP}$ − $\frac{1}{2}$ h. l. $\frac{o}{2AB}$, an infinite quantity, that is, indefinitely greater than the attraction of part of the line AP set off from P.

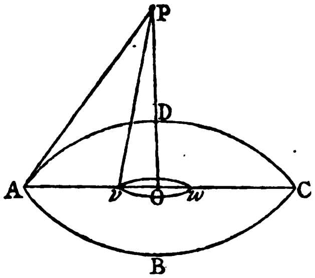
If we take $x=Ap$, the attraction = $\frac{1}{2}$ h. l. $\frac{Bp-AB}{AB+Bp}$ − $\frac{1}{2}$ h. l. $\frac{o}{2AB}$; hence, the attraction of Pp to AB = $\frac{1}{2}$ h. l. $\frac{BP-AB}{AB+BP}$ − $\frac{1}{2}$ h. l. $\frac{Bp-AB}{AB+Bp}$ = h. l. $\sqrt{\frac{BP-AB}{AB+BP} \times \frac{AB+Bp}{Bp-AB}}$.

Prop. XXXIII.

Let O *be the centre of a circle* ABCD, *and a corpuscle* P *be situated in the line* OP *perpendicular to it's plane; to find the attraction of* P *to the* circle, *supposing the attractive force of* P *to every particle of the circle to vary as the* nth *power of the distance.*

(69.) Put $PO = a$, $Pv = x$, $p = 3{,}14159$, &c. then $Ov^2 = x^2 - a^2$, and by Art. 49. $p \times (x^2 - a^2)$ = the area of the circle vw; hence, $2px\dot{x}$ is the fluxion of the area at the distance Ov from the centre; and by the resolution of forces, $x : a :: x^n$ (the attraction of P toward v) : ax^{n-1} the attraction of P to a corpuscle at

v in the direction PO; hence, the fluxion of the at-



traction of P towards the circle is as $2px\dot{x} \times ax^{n-1} = 2pax^{n}\dot{x}$, or as $ax^{n}\dot{x}$, whose fluent is $\frac{ax^{n+1}}{n+1}$; but when $v=a$, $Ov=0$, and consequently the attraction vanishes; but in this case, the fluent is $\frac{a^{n+2}}{n+1}$; therefore the fluent corrected becomes $\frac{ax^{n+1}}{n+1} - \frac{a^{n+2}}{n+1}$; and when $x = PA$ (neglecting the constant denominator) it becomes $PO \times PA^{n+1} - PO^{n+2}$, which is as the whole attraction towards the circle.

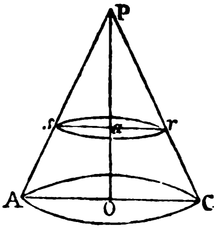
If $n = -2$, it becomes $1 - \frac{PO}{PA}$, the denominator neglected being now $= -1$.

If n be a negative number greater than 1, and the radius AO become infinite, so that PA becomes infinite, then PA being in the denominator, the first term $PO \times PA^{n+1} = 0$, and the attraction is as PO^{n+2}. Hence, if $n = -2$, the attraction becomes unity; therefore the attraction is the same at all distances PO.

Prop. XXXIV.

Let the attractive force of a corpuscle at P *to each particle vary inversely as the square of the distance; to find the attraction of* P *to the* cone PAC.

(70.) By the last article, the attraction of P to the circle sr is as $1-\frac{Pa}{Ps}=1-\frac{PO}{PA}$; the attraction therefore to every section sr is the same; hence, the attraction to the whole cone is as $\left(1-\frac{PO}{PA}\right)\times$ number of sections, or as $\left(1-\frac{PO}{PA}\right)\times PO$, or as $PO-\frac{PO^2}{PA}$.

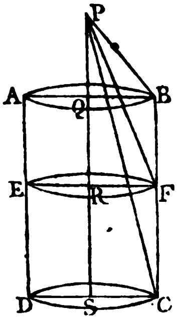


Hence, for *similar* cones, $\frac{PO}{PA}$ being constant, the attraction varies as the length.

Prop. XXXV.

If a corpuscle be situated at P *in the axis* SQ *of a cylinder, to find it's attraction to the* cylinder, *supposing the attractive force to each particle to vary inversely as the square of the distance.*

(71.) Put $RF=a$, $Pr=x$, then $PF=\sqrt{a^2+x^2}$; and by Art. 69. the attraction of P towards the circle EF is as $1-\frac{x}{\sqrt{a^2+x^2}}$; hence, the fluxion of the attractive force is as $\dot{x}-\frac{x\dot{x}}{\sqrt{a^2+x^2}}$, whose fluent is $x-\sqrt{a^2+x^2}$ (Art. 39.); now when $x=PQ$ this fluent becomes $PQ-PB$, and when $x=PS$, it becomes $PS-PC$; and as we want the attraction of P to the solid between these two values of x, their difference $SQ +$



$PB - PC$ is as the attraction required.

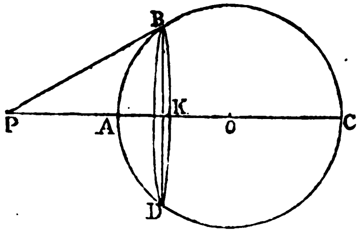
If the length be infinite, then $PC = PS$; therefore $SQ - PC = SQ - PS = -PQ$, and the attraction becomes as $PB - PQ$.

If the diameter AB be infinite, then $PC = PB$; hence, the attraction becomes as SQ.

Prop. XXXVI.

To find the attraction of a corpuscle P *to a* sphere, *when the attraction to each particle varies inversely as the square of the distance; or directly as the distance, whatever be the form of the body.*

(72.) ***First***, let the force vary inversely as the square of the distance. Let O be the centre, $PAOC$ perpendicular to BD; put the radius $AO = a$, $OP = b$, $AP = b - a = c$, $PK = y$, and let $PB = c + x$, then



$AK = y - c$, $CK = 2a - y + c$, $\therefore (y - c) \times (2a - y + c) = BK^2 = BP^2 - PK^2 - \overline{c + x}|^2 - y^2$; hence, $y = \frac{2ac + 2c^2 + 2cx + x^2}{2a + 2c} =$ (as $b = a + c$) $\frac{2bc + 2cx + x^2}{2b}$; therefore the attraction of P to the circle BD is (Art. 69.) as $1 - \frac{2bc + 2cx + x^2}{2b \times (c + x)}$, or as $\frac{2ax - x^2}{b \times (c + x)}$; also, $\dot{y} = \frac{c\dot{x} + x\dot{x}}{b}$; hence, the fluxion of the attraction to the sphere is as $\frac{2ax\dot{x} - x^2\dot{x}}{b^2}$, whose fluent is $\frac{ax^2 - \frac{1}{3}x^3}{b^2}$, the attraction to ABD, for the fluent wants no correction, as it becomes $= 0$ when $ABD = 0$; and when $x = 2a$, it is $\frac{4a^3}{3b^2}$ the attraction to the whole sphere, which therefore varies as $\frac{a^3}{b^2}$.

If the density d of the sphere should vary, then the attraction will vary as $\frac{da^3}{b^2}$.

If the corpuscle be at the surface of the sphere, then $a = b$, and the attraction varies as da; or if d be given, it varies as a.

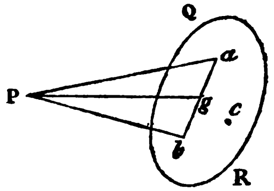
Since the quantity of matter m varies as da^3, the attraction varies as $\frac{m}{b^2}$. Now if the sphere were evanescent in magnitude, with the same quantity of matter, the attraction would be the same, it being independent of a. Hence, *the attraction of a corpuscle to a sphere, is the same as if all the matter of the sphere were collected into it's centre.*

COR. 1. Hence, if P be a sphere of finite magnitude, the attraction of that sphere to every point of $ABCD$ will be the same as if the whole quantity of matter in P were collected into it's centre. Therefore two spheres attract each other by the same force as if the matter of each was collected into it's centre, or by a force varying inversely as the square of the distance of their centers. Now Sir I. Newton has proved, that any two bodies in our system attract each other by forces varying inversely as the squares of the distances of their centers; we therefore conclude conversely, that the constituent particles attract each other by forces which vary inversely as the squares of their distances.

Cor. 2. If P be taken *within* the sphere, in like manner we find the attraction of P to the parts lying between P and A, P and C, and the difference of these attractions will be the whole attraction of P to O, and it comes out to be as the quantity of matter in the sphere of which PO is the radius, divided by PO^2, or as PO. Hence, a corpuscle situated *within* a sphere, is attracted towards the centre by a force varying as the distance from the centre.

Cor. 3. Further, as two distant spheres A, B, attract each other by a force varying inversely as the squares of the distances $\left(\frac{1}{d^2}\right)$ of their centers, if from A there be taken away a concentric sphere, as B attracts what is taken away from A by a force $\frac{1}{d^2}$, B must now attract the shell left, by a force $\frac{1}{d^2}$. Now if in the place of the sphere taken away, there be put another sphere of different density, it will attract that by a force $\frac{1}{d^2}$; therefore it will attract the whole of B as now formed, by a force $\frac{1}{d^2}$. Thus you may take away spheres from B and add others of different densities, so as to make the density of B to vary from the centre to the circumference according to any law. We may reason the same of A. Hence, if two spheres have their densities vary from their centers to their circumferences according to any law, they will attract each other by a force which varies as $\frac{1}{d^2}$. Or two shells will do the same; or a sphere and a shell.

Secondly, Let the force vary directly as the distance of P from every point of the body QR. Conceive the body to be divided into an indefinite number of corpuscles a, b, c, &c. Join ab, and let g be their centre of gravity, and join Pa, Pb, Pg. Now the attraction of P to a is as $A \times Pa$, and to b as $b \times Pb$; resolve the former into $a \times ag$ and $a \times Pg$, and the latter into $b \times bg$ and $b \times Pg$; but $a \times ag = b \times bg$, these attractions therefore being equal and opposite, destroy each other; the whole attraction therefore of P to a and b is $a \times Pg + b \times Pg = (a+b) \times Pg$, and is the same as if a, b, were placed in their centre of gravity. Conceive therefore a and b to be placed at g, and take another particle c; then in like manner the attraction of P to $a+b$ and c is the same as if the quantity of matter of $a+b$ and c were placed in their centre of gravity. Thus we may go on through all the particles. Hence, all the attraction of P to the body QR is the same as if the matter were placed in it's centre of gravity. Hence, as in the former case, if P be of finite magnitude, it's attraction to QR will be the same as if it's quantity of matter were collected into it's centre of gravity. Also, for two spheres we may apply the reasoning in the last article.



In any other law of attraction than the two here mentioned, the attraction of a corpuscle without a sphere to a sphere, will not be the same as if all the matter in the sphere were collected into the centre. Nor is there any fixed point in the sphere, into which, if the whole matter were collected, the attraction would at all distances continue the same as for the sphere.

SECT. VII.

ON SECOND, THIRD, &c. FLUXIONS, AND POINTS OF CONTRARY FLEXURE.

PROP. XXXVII.

To explain under what circumstances a quantity may have several orders of fluxions.

(73.) THE fluxion of a quantity being the uniform increase or decrease of that quantity in a given time, every quantity which increases or decreases must have a fluxion. Hence, if the fluxion of any quantity be not constant, it must have some certain rate of increase or decrease, which rate of increase or decrease will therefore be the fluxion of that fluxion, or the second fluxion of the original flowing quantity. Also, if this second fluxion be not always the same, it must have a rate of variation, that rate therefore will be the fluxion of the second fluxion, or the third fluxion of the original quantity; and so on*. Thus a quantity will have a successive order of fluxions till some one fluxion becomes constant, and then by Art. 3. it will have no more. Thus, let x increase uniformly; then the fluxion of x^2 is $2x\dot{x}$; now $\dot{x}$ is constant, but x itself increases, therefore $2x\dot{x}$ increases in proportion to the increase of x; the fluxion therefore of x^2 is not constant. Hence, considering x as the variable part of $2x\dot{x}$, it's fluxion by Art. 9. is $2\dot{x}\dot{x} = 2\dot{x}^2$, which is the second fluxion of x^2. But if we suppose x not to increase uniformly, then $2x\dot{x}$ will have both x and $\dot{x}$ variable; hence, by Art. 15. the fluxion of $2x\dot{x}$ will be $2\dot{x}\dot{x} + 2x\ddot{x}$, or $2\dot{x}^2 + 2x\ddot{x}$, which therefore is the second fluxion of x^2. But if we should here suppose neither $\dot{x}$ nor $\ddot{x}$ to be constant, then this second fluxion would be variable. Now the fluxion of $2\dot{x}^2$ is found by Art. 13. considering here $\dot{x}$ as the root, and therefore the fluxion of the root is $\ddot{x}$; hence, the fluxion of $2\dot{x}^2$ is $4\dot{x}\ddot{x}$; also, the fluxion of $2x\ddot{x}$ is found by Art. 15. to be $2\dot{x}\ddot{x}+2x\dddot{x}$, both x and $\ddot{x}$ being variable; therefore the fluxion of $2\dot{x}^2 + 2x\ddot{x}$, or the third fluxion of x^2, is $4\dot{x}\ddot{x} + 2\dot{x}\ddot{x} + 2x\dddot{x} = 6\dot{x}\ddot{x} + 2x\dddot{x}$. In like manner we may find the successive orders of fluxions of any quantity.

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* The fluxion of $\dot{x}$ is denoted thus, $\ddot{x}$; the fluxion of $\ddot{x}$ is denoted thus, $\dddot{x}$; and so on.

(74.) If x increase uniformly, or if $\dot{x}$ be constant, x^n will have n fluxions, and no more, n being an affirmative whole number. For the first fluxion is $nx^{n-1}\dot{x}$; and x only being variable, it's fluxion is $n.(n-1).x^{n-2}\dot{x}^2$; and the fluxion of this is $n.(n-1).(n-2).x^{n-3}\dot{x}^3$ &c. when therefore we have taken the fluxion n times, the index of x becomes $= 0$, and $x^0 = 1$; hence, the fluxion then becomes $n.(n-1)\ldots 2.1.\dot{x}^n$, which being a constant quantity, it has no further fluxion.

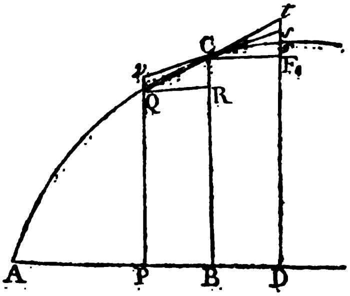
(75.) The first fluxion of x^3+ay^2 is $3x^2\dot{x} + 2ay\dot{y}$; and if $\dot{x}$ and $\dot{y}$ be both variable, it's fluxion is $6x\dot{x}^2 + 3x^2\ddot{x} + 2a\dot{y}^2 + 2ay\ddot{y}$; but if $\dot{x}$ be constant, then $\ddot{x} = 0$; therefore the second fluxion becomes $6x\dot{x}^2 + 2a\dot{y}^2 + 2ay\ddot{y}$; and if $\dot{y}$ be constant, the second fluxion is $6x\dot{x}^2 + 3x^2\ddot{x} + 2a\dot{y}^2$.

(76.) The first fluxion of x^ny^m, by Art. 15. is $ny^mx^{n-1}\dot{x} + mx^ny^{m-1}\dot{y}$; and if both $\dot{x}$ and $\dot{y}$ be variable, we are to consider each of these quantities as composed of three variable factors, and then the fluxion, by the same Art. will be $n.mx^{n-1}y^{m-1}\dot{y}\dot{x} + n.(n-1).y^mx^{n-2}\dot{x}^2 + ny^mx^{n-1}\ddot{x} + m.(m-1).x^ny^{m-2}\dot{y}^2 + mny^{m-1}.x^{n-1}\dot{x}\dot{y} + mx^ny^{m-1}\ddot{y}$.

PROP. XXXVIII.

To find the second fluxion of the ordinate of a curve.

(77.) Let PQ, BC, Dr be three equidistant ordinates, draw QR, CE parallel to AB, and let vCs be a tangent at C, meeting PQ, Dr in v and s; join QC, and



produce it to meet Ds in t. Now as $PB = BD$, the increment of the abscissa is constant, therefore (Art. 3, Cor. 1.) PB or BD will represent the fluxion of the abscissa, which is also constant. Now the cotemporary increments of the ordinates are RC, Er; but the triangles QRC, CEt are similar, and $QR = CE$, therefore $RC = Et$; consequently the cotemporary increments of the ordinates are Et, Er, and their difference is rt; but as the limit of the increment or decrement of the ordinate is the fluxion of the ordinate (Art. 7.), therefore the limit of rt, the difference between two successive increments of the ordinate, or the limit of the increment of the increment, will be the fluxion of the fluxion of the ordinate, or the second fluxion of the ordinate. Now as the triangles CvQ, Cst are similar, and $QC = Ct$, therefore $Qv = st$; and as Qv, sr depend upon the curvature of CQ, Cr, if Q and r be brought up to C, so as to get the measure of the curvature at C from each side, it is manifest that the *limit* of Qv to sr must be a ratio of equality; hence, the *limiting* ratio of rs : st is that of equality; consequently the *limiting* ratio of $rt : 2rs$ is a ratio of equality. Hence, if we take $2rs$ in two different parts of the curve and make them vanish, their *limiting* ratio expresses the ratio of the second fluxions of the ordinates. Moreover, rt expresses the difference between the two successive increments of the ordinates, cotemporary with Er which expresses the difference of the two ordinates themselves; therefore by taking the *limit*, so that the latter increment may become the fluxion of the ordinate, the former becomes the fluxion of the fluxion of the ordinate, or the second fluxion of the ordinate; hence, whilst the *limit* of rt, or $2rs$, expresses the *second* fluxion of the ordinate, the limit of Er will express it's *first* fluxion; but (Art. 23.) the limit of Er is Es the fluxion of the ordinate, CE and Cs expressing the cotemporary fluxions of the abscissa and curve (Art. 27.); therefore the *limits* of $2rs$, Cs and CE, express the cotemporary second fluxion of the ordinate, the fluxion of the curve AC, and the fluxion of the abscissa AB. In like manner it appears, if the curve be a spiral.

ON THE POINT OF CONTRARY FLEXURE OF A CURVE.

DEFINITION.

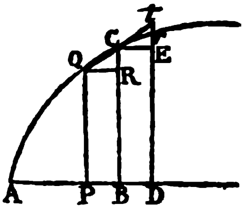
(78.) If a curve be concave in one part and convex in another, the point where the concave part ends and the convex begins, is the point of *contrary flexure.*

PROP. XXXIX.

To find the point of contrary flexure of a curve.

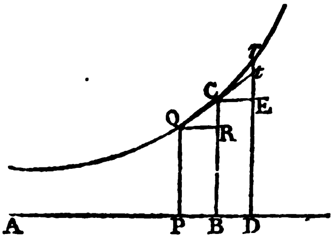
(79.) Let PQ, BC, Dr, be three equidistant ordinates, and the curve *concave* to the axis; and draw

QR, *CE* parallel to *AD*, and join *QC*, and produce it to meet *Dr* in *t*. Then the triangles *QRC*, *CEt*, being similar, and $QR=CE$, therefore $CR=tE$, and



hence *CR* is greater than *Er*; therefore if y represent the ordinate, moving from *A*, and x the abscissa, and $PB=BD=\dot{x}$ a constant quantity; then corresponding to the uniform increase of x, the increment of y, and consequently $\dot{y}$, decreases; now as y increases, $\dot{y}$ is positive by Art. 16. but as $\dot{y}$ decreases, it's fluxion, or $\ddot{y}$, is negative by the same article.

If the curve be *convex* to the axis, and the ordinate move from *A*, then the increment of y, and therefore $\dot{y}$, increases; and as y increases, $\dot{y}$ is positive; and, as $\dot{y}$ increases, it's fluxion, or $\ddot{y}$, is positive. Therefore when the curve is *concave* to the axis, $\ddot{y}$ is *negative*; when *convex*, $\ddot{y}$ is *positive*, $\dot{x}$ being constant. Hence, at the



point of contrary flexure, $\ddot{y}$ changes it's sign; but a quantity may change it's sign, either by passing through 0, or *infinity*;* hence, at the point of contrary flexure, $\ddot{y} = 0$, or *infinity*. What we here mean by infinity is only in respect to it's value at any other time, that term being relative; and in this case we are to understand that $\ddot{y}$ is indefinitely greater at that time than at any other. If we conceive a line to be drawn from *A* parallel to *BC*, and consider it as an abscissa to the curve, and draw lines from it to *Q*, *C*, *r*, parallel to *AD*; then the former abscissæ *AP*, *AB*, *AD* become equal to the ordinates, and the ordinates *PQ*, *BC*, *Dr*, become equal to the abscissæ; if therefore $\dot{y}$ be made constant, $\ddot{x} = 0$, or *infinity*, at the point of contrary flexure. Hence, we have the following

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* When a quantity from positive pass through *o* to negative, it's inverse will pass through infinity to negative, the signs in the latter case being the same as in the former.

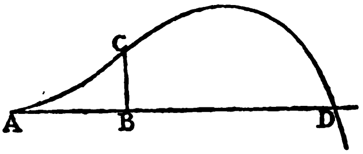
RULE.

Put the equation of the curve into fluxions; make $\dot{x}$ *or* $\dot{y}$ *constant and take the fluxion of the equation again, and get the value of* $\ddot{y}$ *or* $\ddot{x}$, *and put it* = 0, *or infinity; from which find the value of* x, *which gives the abscissa corresponding to the point of contrary flexure. And to determine for any value of* x, *whether the curve be concave or convex, substitute that value for* x *into the expression for* $\ddot{y}$, *the* $\dot{x}$ *being supposed constant, and if it come out* positive, *the curve is* convex *to the axis; if* negative, *it is* concave.

EXAMPLES.

Ex. 1. *Let the equation of the curve be* $y = 3x + 18x^2 - 2x^3$.

Here $\dot{y} = 3\dot{x} + 36x\dot{x} - 6x^2\dot{x}$, and $\ddot{y} = 36\dot{x}^2 - 12x\dot{x}^2$ =(if $\dot{x} = 1$) $36 - 12x$. Now make $36 - 12x = 0$, and



$x = 3$; take therefore *AB* = 3, and draw the ordinate

BC, and *C* is the point of contrary flexure. If x be between 0 and 3, $36-12x$ is positive, therefore the part *AC* of the curve is convex to *AB*; but when x is greater than 3, $36-12x$ is negative, and therefore the curve is concave towards the axis.

Ex. 2. *Let the curve be the Conchoid of* Nichomedes; see Prop. 22, Ex. 7.

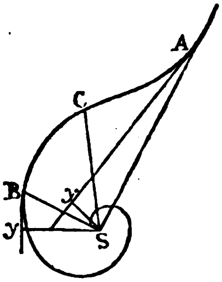
Put $AP = x$, $PM = y$, $CA = a$, $AD = b$; then $xy = (a+y) \times \sqrt{b^2-y^2}$; take the fluxion, and $y\dot{x} + x\dot{y} = \dot{y}\sqrt{b^2-y^2} - \frac{ay\dot{y}+y^2\dot{y}}{\sqrt{b^2-y^2}}$; substitute for x it's value, and we get $\dot{x} = -\frac{y^3+b^2a}{y^2\sqrt{b^2-y^2}} \times \dot{y}$; now make $\dot{y}$ constant, and we have $\ddot{x} = \frac{2b^4a - b^2y^3 - 3b^2ay^2}{(b^2y^3-y^5) \times \sqrt{b^2-y^2}} \times \dot{y}^2$, which put $=0$, in which case the numerator $=0$; hence, $y^3+3ay^2 = 2b^2a$; from whence y may be found, and then x, which will give the point of contrary flexure. This curve is the *Conchoid of Nichomedes.*

Ex. 3. *Let the equation of the curve be* $y = 180x^2 - 110x^3 + 30x^4 - 3x^5$.

Here $\dot{y} = 360x\dot{x} - 330x^2\dot{x} + 120x^3\dot{x} - 15x^4\dot{x}$, and $\ddot{y} = 360\dot{x}^2 - 660x\dot{x}^2 + 360x^2\dot{x}^2 - 60x^3\dot{x}^2 = 0$, or $-x^3 + 6x^2 - 11x + 6 = 0$, whose simple factors are $1-x$, $2-x$, $3-x$, and the roots are 1, 2, 3, the abscissæ corresponding to the points of contrary flexure, of which therefore there are three. As $-x^3 + 6x^2 - 11x + 6 = (1-x) \times (2-x) \times (3-x)$, when x is less than 1, this quantity is positive, and therefore the curve is convex to the axis; when x is between 1 and 2, it is negative, and the curve is concave; when x is between 2 and 3, it is positive, and the curve is convex; when x is greater than 3, it is negative, and the curve will then continue concave.

(80.) If by making $\ddot{y} = 0$, the equation has 2 equal roots, then $\ddot{y}$ passes through 0 without changing it's sign; in this case therefore, the point found is not a point of contrary flexure. And this will always be the case, when the equation has an *even* number of *equal* roots. For more on this subject, see Problems at the end.

(81.) To find the point *C* of contrary flexure of a *Spiral*, it is manifest, that as long as the point *A* approaches to *C*, the perpendicular *Sy* upon the tangent must increase; and after *A* has passed through *C* to *B*, the perpendicular will then decrease; therefore at the point *C* it is a maximum; hence, if we make the fluxion of the perpendicular = 0, it will give the point of contrary flexure.



Ex. *Let the spiral be that in Article* 32.

Here $Sy = \frac{my^{m+1}}{\sqrt{t^{2m} + m^2y^{2m}}}$; hence, $2\,Sy \times \dot{\overline{Sy}} = \frac{2m^4y^{4m+1}\dot{y} + (2m+2) \times m^2t^{2m}y^{2m+1}\dot{y}}{t^{2m} + m^2y^{2m}}$; but $\dot{\overline{Sy}} = 0$, therefore $2m^4y^{4m+1} + (2m+2) \times m^2t^{2m}y^{2m+1} = 0$; hence, $y^{2m} = -\frac{(m+1) \times t^{2m}}{m^2}$, and $y = -\overline{\frac{m+1}{m^2}}\Big|^{\frac{1}{2m}} \times t$. Assuming therefore m a whole number, $2m$ must be an even number, and therefore y is impossible, except m be a negative number greater than 1, in which case the quantity under the radical sign becomes positive.

For the *Lituus*, $m = -2$, and $y = \left.\overline{\frac{1}{4}}\right|^{-\frac{1}{4}} \times t = \left.\overline{4}\right|^{\frac{1}{4}} \times t$ $= \sqrt{2} \times t$.

If $m = 1$, it is the spiral of *Archimedes*, and y is impossible, therefore it has no contrary flexure.

If $m = -1$, it is the *reciprocal* spiral, and y is infinite, therefore it has no contrary flexure.

SECT. VIII.

ON THE MOTION OF BODIES ATTRACTED TO A CENTRE OF FORCE.

PROP. XL.

To find the time and velocity of a body descending or ascending in a non-resisting medium, in a right line to or from a centre of force; supposing the force to vary as any power of the distance from the centre.

(82.) LET v be the velocity of the body at any time, x the corresponding space, either that described, or to be described, $m = 16\frac{1}{12}$ feet, $t =$ the time, F the force compared with the force of gravity on the earth's surface, which we will represent by unity; then $v\dot{v} = \pm 2mF\dot{x}$, the sign being + when v and x increase together, and − when v increases as x decreases. For by *Mechanics*, $\dot{v} \propto F \times \dot{t}$, and $\dot{t} \propto \frac{\dot{x}}{v}$; hence, $\dot{v} \propto F \times \frac{\dot{x}}{v}$, and $v\dot{v} \propto F \times \dot{x}$, that is, $v\dot{v}$ is to $F\dot{x}$ in some constant ratio; let $v\dot{v} = dF\dot{x}$. Now when a body falls upon the earth's surface, $v^2 = 4mx$ by *Mechanics*, x being the space described; hence, $v\dot{v} = 2m\dot{x}$; but if x be the space to be described, and a the whole space, then $v^2 = 4m \times (a - x)$, and $v\dot{v} = -2m\dot{x}$; hence, $v\dot{v} = \pm 2m\dot{x}$; but n this case, $F = 1$; therefore, $d = \pm 2m$; hence, $v\dot{v} = \pm 2mF\dot{x}$. Also, the velocity of a body moving uniformly s measured by the space described in 1″; therefore to find the time corresponding to the space $\pm \dot{x}$, we have $v : \pm\dot{x} :: 1'' : \dot{t} = \pm\frac{\dot{x}}{v}$, because v is the velocity with which $\dot{x}$ is described in the time $\dot{t}$, and when the velocity is uniform, the space is as the time.

Cor. If the force of gravity on the earth's surface be represented by $2m$, then $d = 1$, and $v\dot{v} = \pm F\dot{x}$. If F be constant, $v^2 = \pm 2Fx$. If the forces F, F', F'', &c. act on the body, then $v\dot{v} = \pm(F + F' + F'' + \&c.) \times \dot{x}$.

Prop. XLI.

Let a body begin to fall from any point A *towards the centre of force* S; *to find the velocity at any point* C, *and the time of describing* AC.

(83.) Put $a = SA$, $x = SC$, $v =$ velocity at C, and let

the force vary as x^n, and at any distance c from S, let e represent the force compared with the force of gravity on the earth's surface, or unity; then $c^n : x^n :: e : \frac{e}{c^n} \times x^n = \left(\text{if } d = \frac{e}{c^n}\right) dx^n$, the force at the distance x; hence, $v\dot{v} = -2mdx^n\dot{x}$, and $\frac{v^2}{2} = -\frac{2md}{n+1} \times x^{n+1} + C$; but when $v = 0$, $x = a$, and $0 = -\frac{2md}{n+1} \times a^{n+1} + C$, $\therefore C =$

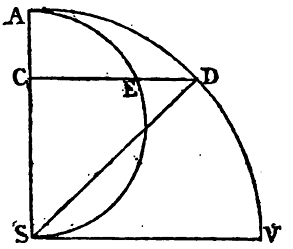
$\frac{2md}{n+1}\times a^{n+1}$; consequently $\frac{v^2}{2}=\frac{2md}{n+1}\times(a^{n+1}-x^{n+1})$, and $v=\sqrt{\frac{4md}{n+1}}\times\sqrt{a^{n+1}-x^{n+1}}$. Hence, $\dot{t} = -\frac{\dot{x}}{v} = -\frac{\dot{x}}{\sqrt{\frac{4md}{n+1}}\times\sqrt{a^{n+1}-x^{n+1}}}$, whose fluent gives t; but this can be found only in particular cases.

Cor. To adapt this to bodies falling at the earth, let c = radius of the earth, $e = 1$.

EXAMPLES.

Ex. 1. If $n = 0$, then $x^n = 1$, and the force is constant, and $v = \sqrt{4md}\times\sqrt{a-x}$. Also, $\dot{t} = \frac{-\dot{x}}{\sqrt{4md}\times\sqrt{a-x}} = \frac{1}{\sqrt{4md}}\times\overline{a-x}|^{-\frac{1}{2}}\times -\dot{x}$, whose fluent (Art 39.) is $t=\frac{2}{\sqrt{4md}}\times\overline{a-x}|^{\frac{1}{2}}+C$; but when $t = 0$, $x = a$, $\therefore C = 0$; hence, $t = \frac{2}{\sqrt{4md}}\times\overline{a-x}|^{\frac{1}{2}}$.

Ex. 2. If $n = 1$, then $v = \sqrt{2md}\times\sqrt{a^2-x^2} = \sqrt{2md}\times CD$, if upon SA a quadrant be described, and the ordinate CD be erected perpendicular to AS.



Also, $\dot{t}=\sqrt{\frac{1}{2md}}\times\frac{-\dot{x}}{\sqrt{a^2-x^2}}$; but if $x = AD$, then

(Art. 46.) $\dot{z} : -\dot{x} :: a : \sqrt{a^2-x^2}$, $\therefore \frac{-\dot{x}}{\sqrt{a^2-x^2}} = \frac{\dot{z}}{a}$; hence, $\dot{t} = \sqrt{\frac{1}{2md}} \times \frac{\dot{z}}{a}$, whose fluent (which wants no correction, because when $t=0$, $z=0$) is $t = \sqrt{\frac{1}{2md}} \times \frac{z}{a}$, the time through AC; hence, if $p = 1{,}57079$ (which is $\frac{1}{4}$ of the circumference of a circle whose radius $= 1$), we have $\sqrt{\frac{1}{2md}} \times p$ for the whole time through AS, because here $z = AV = pa$. Hence, from whatever distance the body falls, the whole time of descent will be the same, it being independent of AS. If $SA = r$, the radius of the earth; the time through $AS = p\sqrt{\frac{r}{2m}}$.

Ex. 3. If $n = -2$, $v = \sqrt{4md} \times \sqrt{x^{-1}-a^{-1}} = \sqrt{4md} \times \sqrt{\frac{a-x}{ax}}$. Also, $\dot{t} = -\frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times \frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{a-x}} = \frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times \frac{-x\dot{x}}{\sqrt{ax-x^2}} = \frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times \left(\frac{\frac{1}{2}a\dot{x} - x\dot{x}}{\sqrt{ax-x^2}} - \frac{\frac{1}{2}a\dot{x}}{\sqrt{ax-x^2}}\right)$, whose fluent (Art. 40. and 46.) is $t = \frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times$ ($\sqrt{ax-x^2}$ − a cir. arc, whose rad. $= \frac{1}{2}a$ and versed sine x) $+ C =$ (if upon AS we describe a semicircle) $\frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times (CE - SE) + C$; but when $t = 0$, this becomes $0 = \frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times -\text{arc } SEA + C$, $\therefore C = \frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times \text{arc } SEA$;

consequently $t = \frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times (CE + \text{arc } AE)$. Hence, the whole time to $S = \frac{a^{\frac{1}{2}}}{\sqrt{4md}} \times \text{arc } AES$.

COR. As the arc AES varies as a, the whole times of descent vary as $a^{\frac{3}{2}}$. Hence, if S be the sun, A the place of a planet at it's mean distance, the times of descent of a planet at it's mean distance to the sun varies in the sesquiplicate ratio of it's major axis, or as the periodic time.

Ex. 4. If $n = -3$, $v = \sqrt{2md} \times \sqrt{x^{-2} - a^{-2}} = \sqrt{2md} \times \frac{\sqrt{a^2 - x^2}}{ax}$. Also, $\dot{t} = \frac{1}{\sqrt{2md}} \times \frac{-ax\dot{x}}{\sqrt{a^2 - x^2}}$, and therefore $t = \frac{1}{\sqrt{2md}} \times a\sqrt{a^2 - x^2} = \frac{1}{\sqrt{2md}} \times AS \times CD$, which wants no correction, because when $t = 0$, $CD = 0$, and both sides vanish together. Hence, the whole time of descent to $S = \frac{1}{\sqrt{2md}} \times AS^2$.

Ex. 5. If $e = 1$, $c = r$, the radius of the Earth, $n = -2$, and a be taken any distance from the Earth's centre greater than r, then $d = r^2$, and $v = \sqrt{4mr^2} \times \sqrt{\frac{a-x}{ax}} = r\sqrt{4m} \times \sqrt{\frac{a-x}{ax}}$ the velocity acquired in falling from any distance a from the centre through $a - x$; and when $x = r$, $v = r\sqrt{4m} \times \sqrt{\frac{a-r}{ar}} = \sqrt{4mr} \times \sqrt{\frac{a-r}{a}}$ the velocity acquired in falling through the space $a - r$ to the Earth's surface.

If a be infinite, $v = \sqrt{4mr}$ the velocity which a body would acquire in falling from an infinite distance.

If $a=2r$, $v=\sqrt{2mr}$ the velocity acquired in falling through a space=radius.

Ex. 6. Let the body fall from the surface of the Earth towards the centre S; then $n=1$, $e=1$, $c=r$, $d=\frac{1}{r}$; hence, $v=\sqrt{\frac{2m}{r}\times(r^2-x^2)}$; and when $x=0$, $v=\sqrt{2mr}$, which is the velocity a body acquires in falling from the surface of the Earth to the centre, because within the Earth's surface the force varies directly as the distance. But by Prop. 148. Cor. 2. this velocity is the velocity of a body revolving about the Earth at it's surface; and $4pr$ being it's circumference, we have $\sqrt{2mr} : 4pr :: 1'' : t''=p\sqrt{\frac{8r}{m}}$ the time in which a body would revolve about the Earth at it's surface. Hence, if SA represent the radius of the Earth; the time of describing $AV=\frac{p}{4}\sqrt{\frac{8r}{m}}=p\sqrt{\frac{r}{2m}}=$ (Ex. 2.) the time of falling from A to the centre S.

Cor. 1. From hence we may find how far a body must fall above the Earth's surface to acquire the velocity in a circle at the surface, supposing $n=-2$; for then, by the two last examples; $\sqrt{4mr}\times\sqrt{\frac{a-r}{a}}=\sqrt{2mr}$; hence, $a=2r$, and $a-r=r$ the space fallen through.

Cor. 2. Let s be the space a body must fall through by the constant force of gravity at the Earth's surface to acquire the velocity $\sqrt{2rm}$ in a circle; then, by *Mechanics*, $v^2=4ms=2rm$; hence, $s=\frac{1}{2}r$; and the same is true for any circle.

Ex. 7. If instead of supposing the body to fall from a state of rest at A, it be projected with a velocity b, then when $x = a$, $v = b$; therefore (Art. 82.) $\frac{b^2}{2} = -\frac{2md}{n+1}\times a^{n+1}+C$; hence, $C=\frac{b^2}{2}+\frac{2md}{n+1}\times a^{n+1}$; consequently $v=\sqrt{b^2+\frac{4md}{n+1}\times(a^{n+1}-x^{n+1})}$. Now to find what height the body will ascend if it be projected upwards, we must put $v = 0$, and then $b^2+\frac{4md}{n+1}\times(a^{n+1}-x^{n+1})=0$; hence, $x=\overline{\frac{n+1}{4md}\times b^2+a^{n+1}}\Big|^{\frac{1}{n+1}}$, the greatest distance from the centre of force to which the body ascends. If we assume $v\dot{v} = \pm F\dot{x}$, we get $v=\sqrt{b^2+\frac{2}{n+1}\times(a^{n+1}-x^{n+1})}$. Here when $v = 0$, $x=\overline{\frac{n+1}{2}\times b^2+a^{n+1}}\Big|^{\frac{1}{n+1}}$ the greatest distance from the centre to which the body can rise; and this never can become infinite so long as the index $\frac{1}{n+1}$ is positive, or as long as n is greater than -1. But when n is less than -1, the index becomes negative, and therefore x is equal to unity divided by $\overline{\frac{n+1}{2}\times b^2+a^{n+1}}\Big|^{\frac{-1}{n+1}}$, which will be finite or infinite according as $\frac{n+1}{2}\times b^2+a^{n+1}$ is positive, or nothing; and if that quantity becomes negative, x becomes negative or impossible, which, as that can never happen, it shows that the supposition of $v=0$ was impossible; that is, the velocity will not be all destroyed when x becomes infinite. If $x=0$, $v=$

$\sqrt{b^2 + \frac{2}{n+1} \times a^{n+1}}$ the velocity at the centre of force when the body is projected downwards. If $b = 0$, or the body fall from a state of rest, $v = \sqrt{\frac{2}{n+1} \times a^{n+1}}$. If $n = 0$, $v = \sqrt{2a}$. If $n = 1$, $v = a$. If n be a greater negative quantity than -1, v comes out impossible, the meaning of which is, that the velocity is greater than can be expressed, even by an infinite quantity.

Ex. 8. If $n = -1$, $v\dot{v} = -2md \times \frac{\dot{x}}{x}$, whose fluent is $\frac{v^2}{2} = -2md \times \text{h. l.}\, x + C$, and when $v = b$, $x = a$, and the fluent becomes $\frac{b^2}{2} = -2md \times \text{h. l.}\, a + C$, and $C = \frac{b^2}{2} + 2md \times \text{h. l.}\, a$; therefore $\frac{v^2}{2} = \frac{b^2}{2} + 2md \times (\text{h. l.}\, a - \text{h. l.}\, x)$, and $v^2 = b^2 + 4md \times \text{h. l.}\, \frac{a}{x}$; hence, $v = \sqrt{b^2 + 4md \times \text{h. l.}\, \frac{a}{x}}$.

It has been here supposed that the body is a point; but when $n = 1$, or -2, the same will be true if the body be spherical, or in the former case, whatever be the form of the body (Prop. 36.). In all other cases, a different method must be pursued to find the time and velocity; see the Problems at the end.

ON THE MOTION OF BODIES IN RESISTING MEDIUMS.

(84.) Let a cylinder move in a fluid in the direction of it's axis, with the velocity d, and suppose the resistance to be equal to the weight of a column of fluid whose base is equal to the end of the cylinder, and altitude k; and let the resistance of a globe of the same diameter as the cylinder, and moving with the same velocity, be to the resistance of the cylinder, as b to 1; these resistances to be determined by experiment, the theory here being defective; and put $p = 0{,}78539$, &c. $h =$ the diameter of the globe, $m = 16\frac{1}{12}$ feet, and let the density of the globe : the density of the fluid :: $n : 1$. Now the magnitude of the globe is $\frac{2}{3}ph^3$, and the magnitude of a column of fluid equal to the resistance of the cylinder is ph^2k; therefore the magnitude of a column of fluid equivalent to the resistance of the globe is pbh^2k. Hence, the magnitude of the globe : magnitude of a column of fluid whose weight = the resistance of the globe $:: \frac{2}{3}ph^3 : pbh^2k :: 1 : \frac{3bk}{2h}$; therefore their quantities of matter are as $n : \frac{3bk}{2h}$, or as $1 : \frac{3bk}{2nh}$.

Hence, if the weight of the globe, or it's gravity, be denoted by unity, $\frac{3bk}{2nh}$ will represent it's resistance moving with the velocity d. Hence, the resistance of the cylinder is $\frac{3k}{2nh}$.

Prop. XLII.

Let a globe be projected in a resisting medium, as in the last article, and let the resistance be as the c^{th} *power of the velocity; to find the velocity* v, *time* t, *and space* x *described, any one in terms of the other.*

(85.) Let d be the velocity of projection, and r the resistance corresponding to the velocity d, compared with the force of gravity represented by unity; then

$r = \frac{3\,b\,k}{2\,n\,h}$ by the last Art. and two values of r corresponding to the two values of d, will give c. Hence, $d^c : v^c :: r : \frac{r}{d^c} \times v^c$ the resistance corresponding to the velocity v; therefore (Art. 82.) $v\dot{v} = -\frac{2\,m\,r}{d^c} \times v^c \dot{x} = \left(\text{if } \frac{1}{e} = \frac{2\,m\,r}{d^c}\right) - \frac{1}{e} v^c \dot{x}$; hence, $\dot{x} = -ev^{1-c}\dot{v}$, consequently $x = -\frac{e}{2-c} \times v^{2-c} + C$; but when $x = 0$, $v = d$, and the equation becomes $0 = -\frac{e}{2-c} \times d^{2-c} + C$; hence, $C = \frac{e}{2-c} \times d^{2-c}$; therefore $x = \frac{e}{2-c} \times (d^{2-c} - v^{2-c})$.

Hence, when $v = 0$, and c is less than 2, $x = \frac{e}{2-c} \times d^{2-c}$, the whole space described before the velocity is all destroyed.

If $c = 2$, $\dot{x} = -\frac{e\dot{v}}{v}$, and $x = e \times \text{h. l.} \frac{d}{v} = \left(\text{because } e = \frac{n\,h\,d^2}{3\,m\,b\,k}\right) \frac{n\,h\,d^2}{3\,m\,b\,k} \times \text{h. l.} \frac{d}{v}$. Hence, when $v = 0$, x becomes infinite, therefore the velocity will never be destroyed.

If c be greater than 2, $2 - c$ is negative, and by making $v = 0$, x becomes infinite, which shows that the velocity will never be all destroyed.

Also (Art. 82.), $\dot{t} = \frac{\dot{x}}{v} = -ev^{-c}\dot{v}$, and $t = -\frac{e}{1-c} \times v^{1-c} + C$; but when $t = 0$, $v = d$; hence, $C = \frac{e}{1-c} \times d^{1-c}$; therefore $t = \frac{e}{1-c} \times (d^{1-c} - v^{1-c})$.

Hence, when $v = 0$, and c is less than 1, $t = \frac{e}{1 - c} \times d^{1-c}$, the time of describing the whole space.

If $c = 1$, $\dot{t} = \frac{-e\dot{v}}{v}$, whose fluent corrected is $t = e \times \text{h. l.} \frac{d}{v}$. Hence, when $v = 0$, t becomes infinite. But it appears from above, that, in this case, the space is finite; hence, the body is an infinite time in describing a finite space, and which space is ed.

If c be greater than 1, then $1 - c$ is negative, and when $v = 0$, t becomes infinite; but the space will still be finite whilst c is less than 2. When c is equal to, or greater than 2, both the space and time will be infinite.

As $v = \overline{d^{2-c} - \frac{2-c}{e} \times x}\Big|^{\frac{1}{2-c}}$, substitute this quantity for v, and it gives $t = \frac{e}{1-c} \times \left(d^{1-c} - \overline{d^{2-c} - \frac{2-c}{e} \times x}\Big|^{\frac{1-c}{2-c}}\right)$, showing the relation between t and x, except in the cases where the fluents fail.

Prop. XLIII.

Let a body be projected in a resisting medium directly to or from a centre of force, and be attracted by a constant force towards that centre; to find the space, time, and velocity.

(86.) Let F be the force compared with gravity which is represented by unity, and retain the notation in Art. 85. Now when the body *descends*, the whole accelerative force $= F -$ the resistance; and when it *ascends*, the retarding force $= F +$ the resistance; that is, in the *former* case the force $= F - \frac{r}{d^c} \times v^c$; and in the *latter*, it $= F + \frac{r}{d^c} \times v^c$. Hence (Art. 82.), $v\dot{v} = \pm 2m \times \left(F \mp \frac{r}{d^c} \times v^c\right) \times \dot{x}$, the *upper* signs being used when the body *descends*, and the *lower* when it *ascends*; hence, (if $\frac{r}{d^c} = e$) $\dot{x} = \frac{1}{2m} \times \frac{\pm v\dot{v}}{F \mp ev^2}$.

If $c = 2$, $\dot{x} = \frac{1}{2m} \times \frac{\pm v\dot{v}}{F \mp ev^2}$, whose fluent (Art. 45.) is $x = \frac{1}{2m} \times \frac{1}{2e} \times -$ h. l. $(F \mp ev^2) + C$; but when $x = 0$, $v = d$, and the fluent becomes $0 = \frac{1}{4me} \times -$ h. l. $(F \mp ed^2) + C$; hence, $C = \frac{1}{4me} \times$ h. l. $(F \mp ed^2)$; consequently $x = \frac{1}{4me} \times$ h. l. $\frac{F \mp ed^2}{F \mp ev^2}$. Hence, we may find v in terms of x; for $4mex =$ h. l. $\frac{F \mp ed^2}{F \mp ev^2}$; therefore put $w =$ the number whose h. l. is $4mex$, and then $w = \frac{F \mp ed^2}{F \mp ev^2}$; hence, $v = \sqrt{\frac{F \mp ed^2 - wF}{\mp we}}$.

(87.) If the body *ascend*, and $v = 0$, $x = \frac{1}{4me} \times$ h. l. $\frac{F + ed^2}{F}$ the distance to which it ascends.

(88.) Let the body *descend.* Now when $F = \frac{rv^2}{d^2}$, the resistance becomes equal to the accelerating force; hence, $v^2 = \frac{F \times d^2}{r}$, and $v = d\sqrt{\frac{F}{r}}$, the greatest velocity the body can acquire; for when the resistance becomes equal to the attractive force, there can be no further acceleration.

(89.) If $d = 0$, $x = \frac{1}{4me} \times$ h. l. $\frac{F}{F \mp ev^2}$.

(90.) Also (Art 82.), $\dot{t} = \frac{\dot{x}}{v} = \frac{1}{2m} \times \frac{\pm \dot{v}}{F \mp ev^2}$; hence, when the body *descends*, $\dot{t} = \frac{1}{2me} \times \frac{\dot{v}}{\frac{F}{e} - v^2}$, whose fluent (Art. 45.), (putting $\frac{F}{e} = a^2$) is $t = \frac{1}{4mae} \times$ h. l. $\frac{a+v}{a-v} + C$; but when $t = 0$, $v = d$, and we get $0 = \frac{1}{4mae} \times$ h. l. $\frac{a+d}{a-d} + C$; hence, $C = -\frac{1}{4mae} \times$ h. l. $\frac{a+d}{a-d}$; consequently $t = \frac{1}{4mae} \times \left(\text{h. l. } \frac{a+v}{a-v} - \text{h. l. } \frac{a+d}{a-d}\right)$. Hence, if we substitute the value of v in terms of x, we shall get t in terms of x. If the body fall from a state of rest, $t = \frac{1}{4mae} \times$ h. l. $\frac{a+v}{a-v}$.

(91.) When the body *ascends*, $\dot{t} = \frac{1}{2m} \times \frac{-\dot{v}}{F + ev^2} = \frac{1}{2me} \times \frac{-\dot{v}}{a^2 + v^2}$, whose fluent (Art. 46.) is $t = \frac{1}{2mea} \times -M + C$, M being a circular arc whose radius is

1, and tangent $\frac{v}{a}$; but when $t = 0$, $v = d$; put therefore $N =$ the arc whose tangent is $\frac{d}{a}$, and we get $t = \frac{1}{2mea} \times (N - M)$. For the whole ascent, $v = 0$, $\therefore M = 0$; hence, $t = \frac{1}{2mea} \times N$.

(92.) If we apply these expressions to the descent of a globe in resisting mediums upon the earth's surface, then as unity represents the force of gravity, that is the force when a body falls in vacuo, we must find the value of F when a body descends in the medium. Let the density of the body : the density of the medium :: $n : 1$; then if $w =$ the weight of the body in vacuo, we have, by *Hydrostatics*, w : weight lost when in the fluid :: $n : 1$; hence, $w : w -$ weight lost, or weight in the fluid, :: $n : n - 1$, therefore the weight in the fluid $= w \times \frac{n-1}{n} =$ (if $w = 1$ the force of gravity) $\frac{n-1}{n}$ which is the gravity of the body in the fluid, or the force with which it endeavours to descend; this therefore is the value of F. Also, $c = 2$.

(93.) By Art. 84. $r = \frac{3bk}{2nh}$; hence, (Art. 88.) $v \left(= d\sqrt{\frac{F}{r}} = a\right) = d\sqrt{\frac{2.(n-1).h}{3bk}}$, the greatest velocity the body can acquire by falling in the fluid. Also, $t = \frac{1}{4mae} \times \left(\text{h.l.}\frac{a+v}{a-v} - \text{h.l.}\frac{a+d}{a-d}\right)$; and when $v = a$, t becomes infinite; therefore the body never can acquire it's greatest velocity.

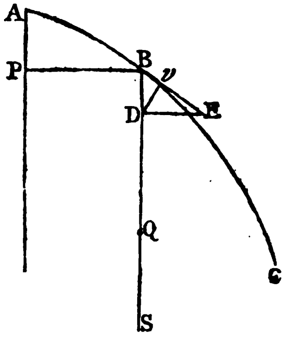
.) The greatest height to which a body can as- when projected upwards, is (Art. 87.) $\frac{1}{4me} \times$ h. l.

$$\frac{d^2}{} = \frac{nd^2h}{6mbk} \times \text{h. l.} \left(1 + \frac{3bk}{2.(n-1).h}\right).$$

Prop. XLIV.

determine the resistance of a medium, by which y may describe any curve about a centre of force, rce to the centre being given.

.) Let ABC be the given curve, S the centre of and F the force of the body at B towards it, rce of gravity being unity; draw DE perpendi- to BS, meeting the tangent BE; and Dv per- ular to BE. Put $AB = z$, $BS = w$, $BD = -\dot{w}$,



: $\dot{z}$, $v =$ the velocity in the curve at B, and $s =$ ½ the chord of the circle of curvature at B passing h S, $m = 16\frac{1}{12}$ feet. Now it is well known, that y, whether it moves in a resisting medium, or nust fall down ¼ s by the constant force F to e the velocity in the curve; for the resistance no deviation from the tangent, but only retards otion of the body, so that it may preserve it's proper proportion corresponding to the force; hence, by *Mechanics*, $v^2 = 4mF \times \frac{1}{2}s = 2mFs$; therefore $\dot{v} = m \times \frac{F\dot{s} + s\dot{F}}{\sqrt{2mFs}}$ the *whole* fluxion of velocity in the direction BE. But, by *Mechanics*, the velocity V which the force F continuing constant for any time t, would generate in the direction BS, is $2mFt$; $\therefore \dot{V} = 2mF\dot{t} = \left(\text{because } \dot{t} = \frac{\dot{BE}}{v} = \frac{\dot{z}}{v}\right) m \times \frac{2F\dot{z}}{\sqrt{2mFs}}$ the fluxion of the velocity in the direction BS, arising from the force F; hence, $BD : Bv$ ($:: \dot{BE} = \dot{z} : \dot{BD} = -\dot{w}$) $:: m \times \frac{2F\dot{z}}{\sqrt{2mFs}} : m \times \frac{-2F\dot{w}}{\sqrt{2mFs}}$ the fluxion of velocity in the direction BE arising from the force F; from which if we take the whole fluxion $m \times \frac{F\dot{s} + s\dot{F}}{\sqrt{2mFs}}$, there will remain $- m \times \frac{F\dot{s} + s\dot{F} + 2F\dot{w}}{\sqrt{2mFs}}$ which is the fluxion of velocity arising from the resistance, in the time that the force F would generate the fluxion of velocity $m \times \frac{2F\dot{z}}{\sqrt{2mFs}}$; but the fluxion of velocity generated or destroyed in the same time is as the force; hence, the resistance : force $F ::$ $m \times \frac{F\dot{s} + s\dot{F} + 2F\dot{w}}{\sqrt{2mFs}} : m \times \frac{2F\dot{z}}{\sqrt{2mFs}} :: \frac{F\dot{s} + s\dot{F} + 2F\dot{w}}{3F\dot{z}} : 1$, omitting the sign $-$ before the first term, as it only signifies the force to be retarding.

(96.) When the centre S is at an infinite distance, and the force F becomes constant and acts in parallel lines, then $\dot{F} = 0$, and the resistance : force $F :: \frac{\dot{s} + 2\dot{w}}{2\dot{z}}$

: 1. But if we draw AP parallel to BS, and PB perpendicular to it, and put $AP=x$, then $\dot{w}=-\dot{x}$; hence, the resistance : force F :: $\frac{\dot{s}-2\dot{x}}{2\dot{z}}$: 1. Or to obtain this proportion in terms of the abscissa and curve, put $y=PB$; then by Art. 54. $\dot{z}^2=\dot{x}^2+\dot{y}^2$; and by Art. 98. $s=\frac{\dot{z}^2}{\ddot{x}}=\frac{\dot{x}^2+\dot{y}^2}{\ddot{x}}$; therefore if we suppose $\dot{y}$ constant, we shall have $\dot{s}=\frac{2\dot{x}\ddot{x}^2-(\dot{x}^2+\dot{y}^2)\times\dddot{x}}{\ddot{x}^2}=\frac{2\dot{x}\ddot{x}^2-\dot{z}^2\dddot{x}}{\ddot{x}^2}$; hence, $\frac{\dot{s}-2\dot{x}}{2\dot{z}}=-\frac{\dot{z}\dddot{x}}{2\ddot{x}^2}$; therefore the resistance : force F :: $\frac{\dot{z}\dddot{x}}{2\ddot{x}^2}$: 1.

EXAMPLES.

Ex. 1. *Let the curve be a* parabola, *and the force be constant and act in lines parallel to* AP.

Put $x=AP$, $y=PB$, then $ax=y^n$, $\therefore a\dot{x}=ny^{n-1}\dot{y}$, and ($\dot{y}$ being constant) $a\ddot{x}=n.(n-1).y^{n-2}\dot{y}^2$; also, $\dddot{x}=\frac{n.(n-1).(n-2)}{a}\times y^{n-3}\dot{y}^3$, and $\dot{z}=\frac{\overline{n^2y^{2n-2}+a^2}|^{\frac{1}{2}}\times\dot{y}}{a}$;

hence, $\frac{\dot{z}\dddot{x}}{2\ddot{x}^2}=\frac{n-2}{2.n.(n-1)}\times\frac{\overline{n^2y^{2n-2}+a^2}|^{\frac{1}{2}}}{y^{n-1}}$, the resistance.

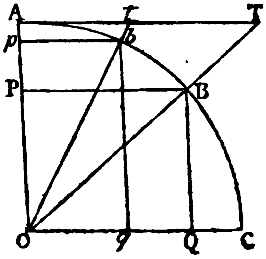
If $n=2$, the resistance becomes $=0$.

If n be less than 2, but greater than 1, the resistance becomes negative; the medium therefore must propel the body, not retard it.

If $n=1$, the medium becomes an infinite propelling one, and the body moves in a right line.

Ex. 2. *Let* ABC *be a* quadrant of a circle, *and the force be constant and act parallel to* AO.

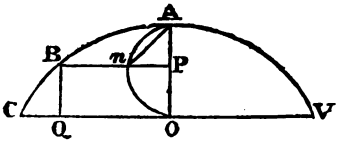
Put $AO=a$, $AP=x$, $AB=z$, then $BQ=s=a-x$, and $\dot s=-\dot x$; hence, $\frac{\dot s-2\dot x}{2\dot z}=-\frac{3\dot x}{2\dot z}=-\frac{3PB}{2OB}$ = the resistance, gravity being unity. Hence, at A the resistance $=0$. When $3PB=2BO$, or radius : sine of AB :: 3 : 2, the resistance = gravity; and at C the resistance : gravity :: 3 : 2. Also, the velocity is as $\sqrt{BQ}$. Hence also, the resistance at $B \propto PB$. Now if we suppose the resistance to vary as the density of the medium × the square of the velocity, then the density varies as the resistance directly and square of the velocity inversely, or as $\frac{PB}{BQ}=\frac{PB}{PO}=\frac{AT}{AO}$; hence, the density at B varies as the tangent of AB. All this agrees with what Sir I. NEWTON has proved in his *Principia*, *Lib.* 2. *Sec.* 2. *Pr.* 10.



Ex. 3. *Let* CAV *be a* cycloid, *and the force be constant and act perpendicularly to the base* CV.

Here $BQ=\frac{1}{2}s$, and if $AO=a$, $\frac{1}{2}s=a-x$, therefore $\frac{1}{2}\dot s=-\dot x$, and $\dot s=-2\dot x$; also, $\dot z=\frac{a^{\frac{1}{2}}\dot x}{x^{\frac{1}{2}}}$ (Art. 54. Ex. 2.);

hence, $\frac{\dot{s}-2\dot{x}}{2\dot{z}}=\frac{-2x^{\frac{1}{2}}}{a^{\frac{1}{2}}}$ = (because $x : An :: An : AO$



$=a)\ \frac{2An}{AO}$ the resistance, gravity being unity. Also, the velocity varies as $\sqrt{BQ}$.

Ex. 4. *Let the force tend to a centre* S, *and vary as* w^n, *and the curve be the* logarithmic spiral.

As $F=w^n$, $\dot{F}=nw^{n-1}\dot{w}$; also, $s=w$, $\therefore \dot{s}=\dot{w}$; hence, the resistance $=\frac{w^n\dot{w}+nw^n\dot{w}+2w^n\dot{w}}{2w^n\dot{z}}=\frac{n+3}{2}\times\frac{\dot{w}}{\dot{z}}=$ (as $\dot{w} : \dot{z}$ in some constant ratio $c : d$) $\frac{n+3}{2}\times\frac{c}{d}$, the force tending to S being unity.

If $n=-3$, the resistance $=0$.

If $n+3$ be negative, the medium must propel the body.

Also, $v=\sqrt{2mFs}=\sqrt{2m}\times w^{\frac{n+1}{2}}$. Now the resistance being to the force F, as $\frac{n+3}{2}\times\frac{c}{d}$ to 1, if F be represented by it's true value w^n, the resistance will become $\frac{n+3}{2}\times\frac{c}{d}\times w^n$; and since the density of the medium varies as the resistance directly and the square of the velocity inversely, the density varies as $\frac{w^n}{w^{n+1}}$, or as $\frac{1}{w}$. Hence, if the density of the medium vary inversely as the distance, the body may describe the logarithmic spiral, whatever be the value of n; agreeably to what Sir I. NEWTON has proved in his *Principia*, *Lib.* 2. *Sec.* 4. *Prop.* 16. If $n = -2$, $F = \frac{1}{w^2}$, or F varies as the square of the density, as he has also proved in *Prop.* 15.

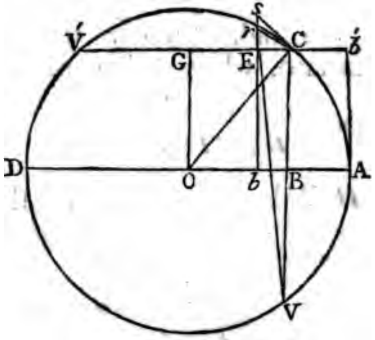
SECT. IX.

ON THE RADIUS OF CURVATURE, AND THE EVOLUTE OF CURVES.

PROP. XLV.

To find the radius of a circle in terms of the fluxions of it's abscissa, ordinate and curve.

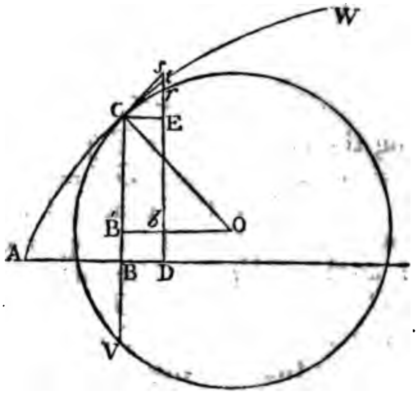
(97.) LET $ACrDV$ be a circle, O the centre, CBV perpendicular to AD, Cs a tangent at C, brs parallel to CB, cutting the curve at r, and join rC, rV. Put $AB = x$, $BC = y$, $AC = z$, and $OC = a$, then $Cs = \dot{z}$, $CE = \dot{x}$, $Es = \dot{y}$. Now the triangles Crs, CVr are similar, for the angle $srC =$ alter. ang. rCV, and the angle $sCr =$ angle CVr in the alternate segment; hence, $sr : rC :: rC : CV = 2CB$; but by Art. 23. it appears that the *limiting* ratio of $rC : sC$ is a ratio of equality; therefore the *limiting* ratio of $sr : rC$ is sr :



sC, or (Art 77.) $-\frac{1}{2}\ddot{y} : \dot{z}$, the sign $-$ being prefixed, for the reason in Art. 78. the curve being concave to the axis; hence $-\frac{1}{2}\ddot{y} : \dot{z} :: \dot{z} : 2BC$, $\therefore BC = \frac{\dot{z}^2}{-\ddot{y}}$; and by similar triangles CEs, CBO, $\dot{x} : \dot{z} :: \frac{\dot{z}^2}{-\ddot{y}} : CO = \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, $\dot{x}$ being constant. If Ab' be perpendicular to AO, and $b'C$ to Ab'; then considering Ab' as the abscissa and $b'C$ the ordinate, we have, for the same reason, $CO = \frac{\dot{z}^3}{\dot{y}\ddot{x}}$, $\dot{y}$ being constant, and $\ddot{x}$ positive (Art. 79.), the curve being convex to the axis. Lastly, by similar triangles OBC, CEs, $\dot{x} : \dot{z} :: y : r = \frac{y\dot{z}}{\dot{x}}$, and if we make $\dot{z}$ constant, we have $\frac{\dot{x}\dot{y}\dot{z} - y\ddot{x}\dot{z}}{\dot{x}^2} = 0$; hence, $y = \frac{\dot{x}\dot{y}}{\ddot{x}}$; and by the same proportion, $\dot{x} : \dot{z} :: y\left(\frac{\dot{x}\dot{y}}{\ddot{x}}\right) : r = \frac{\dot{y}\dot{z}}{\ddot{x}}$. Thus we get the radius under three circumstances, when $\dot{x}$ is constant, when $\dot{y}$ is constant, and when $\dot{z}$ is constant.

DEFINITION.

(98.) Let ACW be any curve, AB the abscissa, BC



the ordinate, *Cs* a tangent at *C*, and let *O* be the centre of a circle touching the curve in *C*, and draw *OB'* parallel to *AB*, and *DbErts* parallel to *BC*, cutting the curve in *t* and the circle in *r*; then if, by bringing *Ds* up to *BC*, the *limiting* ratio of *sr* : *st* be a ratio of equality, the circle is said to be a *circle of curvature to the curve.*

Prop. XLVI.

To find the radius OC *of the circle of curvature to the curve* AC *at the point* C.

(99.) Whether we regard the curve *AC* or the circle, *CE*, *Es*, *Cs* will be the first fluxions of the abscissa, ordinate, and curve; for (Art. 23.) these fluxions depend entirely upon the position of the tangent, which is common to both; and by the Def. (Art. 98.) the *limiting* ratio of *sr* : *st* being a ratio of equality, the second fluxions of the ordinates are equal (Art. 77.); hence, the second fluxion of the ordinate is the same, whether we regard the curve or circle. Now in the *circle*, if x, y, and z represent the abscissa, ordinate, and curve, $CO = \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$ (Art. 97.), $\dot{x}$ being constant; hence, in the *curve AW*, if x, y and z represent the abscissa *AB*, ordinate *BC*, and curve *AC*, the radius of curvature $CO \pm \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$. For the same reason, $CO = \frac{\dot{z}^3}{\dot{y}\ddot{x}}$, when $\dot{y}$ is constant; and $CO = \frac{\dot{y}\dot{z}}{\ddot{x}}$, when $\dot{z}$ is constant. But as $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$, we have $2\dot{x}\ddot{x} + 2\dot{y}\ddot{y} = 0$, and $\frac{\dot{y}}{\ddot{x}} = -\frac{\dot{x}}{\ddot{y}}$; hence, $CO = \frac{\dot{x}\dot{z}}{-\ddot{y}}$.

When we make $\dot{x}$, $\dot{y}$ or $\dot{z}$ constant, it will simplify the operation, if we substitute unity for them.

EXAMPLES.

Ex. 1. *Let* AC *be the common* parabola; *to find the radius of curvature.*

Here $ax=y^2$, $\therefore\ y=a^{\frac{1}{2}}x^{\frac{1}{2}}$, and $\dot{y}=\frac{1}{2}a^{\frac{1}{2}}x^{-\frac{1}{2}}$, $\dot{x}$ being constant and $=1$; hence, $\ddot{y}=-\frac{1}{4}a^{\frac{1}{2}}x^{-\frac{3}{2}}=-\frac{a^{\frac{1}{2}}}{4x^{\frac{3}{2}}}$; also, $\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}=\sqrt{1+\frac{a}{4x}}=\frac{1}{2}\sqrt{\frac{4x+a}{a}}$; therefore CO $=\frac{\dot{z}^3}{-\dot{x}\ddot{y}}=\frac{\overline{4x+a}\rvert^{\frac{3}{2}}}{2\sqrt{a}}$.

When $x=0$, $CO=\frac{1}{2}a$, the radius of curvature at the vertex.

Ex. 2. *Let the curve be an* ellipse.

By Prop. 10. Ex. 2. if $m^2=\frac{b^2}{a^2}$, $y^2=m^2\times(a^2-x^2)$; hence (making $\dot{x}=1$), $y\dot{y}=-m^2x$, $y\ddot{y}+\dot{y}^2=-m^2$, and $-\ddot{y}=\frac{\dot{y}^2+m^2}{y}$; but $\dot{y}=\frac{-m^2x}{y}$; hence, $-\ddot{y}=\frac{m^4x^2+m^2y^2}{y^3}$; also, $\dot{z}^2=1+\frac{m^4x^2}{y^2}=\frac{y^2+m^4x^2}{y^2}$; hence, by substitution and reduction, $\frac{\dot{z}^3}{-\dot{x}\ddot{y}}=\frac{\overline{a^4-(a^2-b^2)\times x^2}\rvert^{\frac{3}{2}}}{ba^4}$ the radius of curvature.

Ex. 3. *Let the curve be the* catenary.

Here (Prop. 130.) $z^2=2ax+x^2$, and putting $\dot{z}$ constant and $=1$, we have $z=a\dot{x}+x\dot{x}$, and $1=a\ddot{x}+x\ddot{x}+\dot{x}^2$; hence, $\dot{x}=\frac{z}{a+x}$, and $\ddot{x}=\frac{1-\dot{x}^2}{a+x}=$

$\frac{(a+x)^2-z^2}{\overline{a+x}|^3} = \frac{a^2}{\overline{a+x}|^3}$; also, $\dot{y} = \frac{a}{a+x}$; hence, $\frac{\dot{y}\,\dot{z}}{\ddot{x}} = a + \frac{z^2}{a}$ the radius of curvature.

At the lowest point $z = 0$, and the radius $= a$.

If $x = y^n$, then $CO = \frac{\overline{1+n^2y^{2n-2}}|^{\frac{3}{2}}}{n.(n-1).y^{n-2}}$. Hence, except for the conic parabola, the radius of curvature is nothing or infinite, according as n is less or greater than 2.

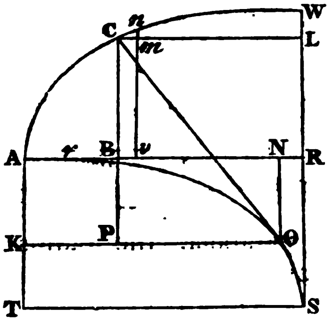
Ex. 4. *Let it be the* logarithmic *curve; to find the radius of curvature.*

By Art. 44. $\dot{y} = \frac{y\dot{x}}{m} =$ (if $\dot{x}$ be supposed constant and $=1$) $\frac{y}{m}$, $\therefore \ddot{y} = \frac{\dot{y}}{m}$, and $-\dot{x}\ddot{y} = -\frac{\dot{y}}{m} = \frac{y}{m^2}$; also, $\dot{z} = \sqrt{\dot{x}^2+\dot{y}^2} = \sqrt{1+\frac{y^2}{m^2}} = \frac{\overline{m^2+y^2}|^{\frac{1}{2}}}{m}$; hence, $CO = \frac{\dot{z}^3}{-\dot{x}\ddot{y}} = \frac{\overline{m^2+y^2}|^{\frac{3}{2}}}{-my}$, which being negative, shows that the centre O lies on the other side of the curve, the curve being concave the other way.

Ex. 5. *Let the curve* ACW *be a* cycloid.

Let WR be the axis, and draw CL parallel to AR; put $ACW = 2WR = a$, $AB = x$, $BC = y$, $AC = z$, then $WC = a - z$, $WL = \frac{1}{2}a - y$; and by the property of the curve, $a^2 (AW^2) : \overline{a-z}|^2 (WC^2) :: \frac{1}{2}a (WR) : \frac{1}{2}(a-y)$ (WL); hence, $y = \frac{2az - z^2}{2a}$, $\dot{y} = \frac{a\dot{z} - z\dot{z}}{a}$, therefore

$\dot{x}^2 = \dot{z}^2 - \dot{y}^2 = \frac{2az - z^2}{a^2} \times \dot{x}^2$, and $\dot{x} = \frac{\overline{2az - z^2}|^{\frac{1}{2}} \times \dot{z}}{a}$, and



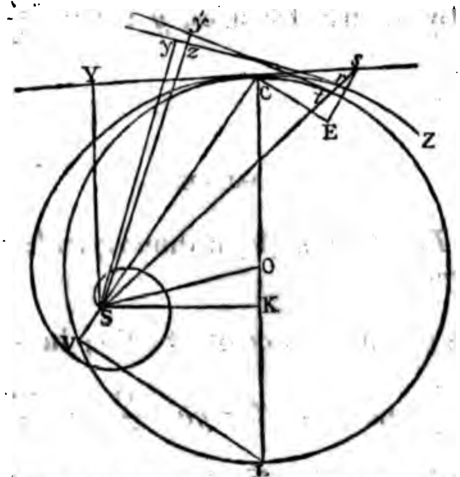
(making $\dot{z}$ constant) $\ddot{x} = \frac{(a - z) \times \dot{z}^2}{a\sqrt{2az - z^2}}$; hence (Art. 99.) $CO \left(= \frac{\dot{y}\dot{z}}{\ddot{x}}\right) = \sqrt{2az - z^2}$.

Cor. When $z = 0$, $CO = 0$; when C comes to W, or $z = a$, then CO (then becoming WS) $= a$; hence, $RS = \frac{1}{2}a$.

TO FIND THE RADIUS OF CURVATURE TO SPIRALS.

(100.) Let CO be the radius of the circle of curvature to the spiral SCZ at C, and draw $Strs$ meeting the tangent YC in s; then by the Definition (Art. 98.), the *limiting* ratio of $sr : st$ is a ratio of equality, consequently rt ultimately vanishes in respect to sr or st. Hence, the tangents ry, ty' will ultimately form with each other an angle which becomes evanescent in respect to the angle formed by the tangents ry and sCY; therefore, ultimately, the difference yy' of the perpendiculars upon the tangents at r and t becomes evanescent in respect to the difference between SY and Sy; consequently the *limit* of the ratios of Sy and Sy' to

SY, must be the same; but the difference between SY



and Sy', SY and Sy, or the increment of SY in each case, is ultimately the fluxion of SY in each case; hence, the fluxion of the perpendicular to a tangent to the curve, and to the circle of curvature, is the same.

Prop. XLVII.

To find the radius OC *of the circle of curvature to the spiral at the point* C.

(101.) Put $SC = y$, draw SK perpendicular to CO, and let $SY = CK = v$, $CO = r$; and considering the point C as describing the circle, the points S and O being fixed, SO is constant; now $OS^2 = OC^2 + CS^2 - 2OC \times CK = r^2 + y^2 - 2rv$, whose fluxion therefore is $= 0$, or $2y\dot{y} - 2r\dot{v} = 0$, r being constant; hence, $r = \frac{y\dot{y}}{\dot{v}}$. Now if we consider y and v in reference to the spiral instead of the circle, $\dot{y}$, or sE, will be the same for each, by Art. 31. because sE depends only upon the position of the tangent; and (Art. 100.) $\dot{v}$ is the same for the circle and spiral; hence, if we consider the point C as describing the spiral, we shall still have $r = \frac{y\dot{y}}{\dot{v}}$

COR. By similar triangles, $y : v :: \frac{2y\dot{y}}{\dot{v}}\ (CL) : CV = \frac{2v\dot{y}}{\dot{v}}$.

EXAMPLES.

Ex. 1. *Let it be the* logarithmic *spiral; to find the radius of curvature.*

Here $y : v :: m : n$, a constant ratio; hence, $v = \frac{ny}{m}$, and $\dot{v} = \frac{n\dot{y}}{m}$; therefore $CO = y\dot{y} \times \frac{m}{n\dot{y}} = \frac{my}{n}$.

Hence, the chord CV of the circle of curvature passing through S, $= \frac{2v\dot{y}}{\dot{v}} = 2y = 2SC$.

Ex. 2. *Let it be the spiral of* Archimedes; *to find the radius of curvature.*

By Art. 32. $v = \frac{y^2}{\sqrt{y^2+t^2}}$; hence, $\dot{v} = 2y\dot{y} \times \overline{y^2+t^2}|^{-\frac{1}{2}} - y\dot{y} \times \overline{y^2+t^2}|^{-\frac{3}{2}} \times y^2 = \frac{2y\dot{y}}{\overline{y^2+t^2}|^{\frac{1}{2}}} - \frac{y^3\dot{y}}{\overline{y^2+t^2}|^{\frac{3}{2}}} = \frac{2y\dot{y} \times (y^2+t^2) - y^3\dot{y}}{\overline{y^2+t^2}|^{\frac{3}{2}}} = \frac{y^3\dot{y} + 2t^2y\dot{y}}{\overline{y^2+t^2}|^{\frac{3}{2}}}$; therefore $CO = y\dot{y} \times \frac{\overline{y^2+t^2}|^{\frac{3}{2}}}{y^3\dot{y} + 2t^2y\dot{y}} = \frac{\overline{y^2+t^2}|^{\frac{3}{2}}}{y^2 + 2t^2}$.

The same expression for the radius of curvature will do for all curves, where the relation between SY and SC is known.

For example, let the curve be a *parabola*, S the focus, and $a = \frac{1}{4}$ of the principal latus rectum; then $y = \frac{v^2}{a}$, and $y^2 = \frac{v^4}{a^2}$, $\therefore y\dot{y} = \frac{2v^3\dot{v}}{a^2}$; hence, $CO = \frac{y\dot{y}}{\dot{v}} = \frac{2v^3}{a^2}$.

Also, $CV = \frac{2v\dot{y}}{\dot{v}} = 4y = 4CS$.

ON THE EVOLUTIONS OF CURVES.

DEFINITION.

If (fig. last but one) a thread *COS* be wound on the curve *rOS*, and by unwinding it, keeping it stretched out, the end *C* describes the curve *ACW*, then *rOS* is said to be the *evolute* to *ACW*, called the *involute*.

Cor. *CO* is always a tangent to *rOS*.

PROP. XLVIII.

(102.) *To find the abscissa and ordinate of the evolute.*

Draw *ON* perpendicular to *AR*, *AT* perpendicular to *AR*, *OK* parallel to *AR*; produce *CB* to meet *KO* in *P*; draw *nv* parallel to *CB*, and *Cm* to *AR*, and let the evolute begin at *r*; put $AB = x$, $BC = y$, $AC = z$, then, (considering *Cn* as a tangent at *n*) $Cn = \dot{z}$, $Cm = \dot{x}$, $mn = \dot{y}$; let $AN = u$, $NO = v$, $CO = r$. Then by sim. tri. $\dot{z} : \dot{y} :: r : OP = r \times \frac{\dot{y}}{\dot{z}}$, $\dot{z} : \dot{x} :: r : CP = r \times \frac{\dot{x}}{\dot{z}}$. Now $AN = AB + OP = x + r \times \frac{\dot{y}}{\dot{z}}$, $rN = x + r \times \frac{\dot{x}}{\dot{z}} - AP$, and $NO = CP - CB = r \times \frac{\dot{x}}{\dot{z}} - y$. Comparing therefore these values of *AN*, *NO*, or *rN*, *NO*, we get the relation of the abscissa and ordinate of the evolute.

Ex. 1. *Let* ACW *be the common* parabola, AR *the axis.*

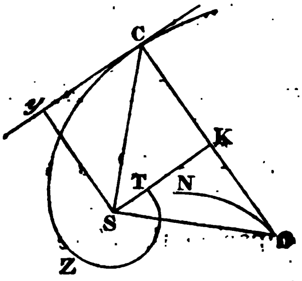
Here (Art. 99.) $r = \frac{\dot{y}\dot{z}}{\ddot{x}} = \frac{\overline{4x+a}|^{\frac{3}{2}}}{2a^{\frac{1}{2}}}$, and at A, $x = 0$, therefore $rA = \frac{1}{2}a$. Also $AN = \left(x + \frac{\dot{y}\dot{z}^2}{-\dot{x}\ddot{y}}\right) = \frac{1}{2}a + 3x$, and $rN = 3x$; also, $NO \left(= \frac{\dot{z}^2}{\dot{y}} - y\right) = \frac{4x^{\frac{3}{2}}}{a^{\frac{1}{2}}}$: hence, $ON^2 \times a : rN^3 :: 16 : 27$ a given ratio; therefore rOS is a semicubical parabola.

Ex. 2. *Let* ACW *be a cycloid*, AR *the axis.*

Produce WR to meet the evolute in S, and draw ST parallel to TS. Then (Art. 101. Ex. 2. Cor.) r coincides with A; and by the same Example, AS^2 $(a^2) : AO^2$ $(2az - z^2) :: AT$ $(= RS = \frac{1}{2}a) : AK$ $\left(= NO = \frac{2az - z^2}{2a}\right)$, which is the property of the cycloid. Hence, SOA is a cycloid similar and equal to ACW.

Ex. 3. *Let* TZC *be the* logarithmic spiral, S *the centre.*

Draw the tangent Cy, on which let fall the perpendicular Sy; let CO be the radius of curvature, on which let fall the perpendicular SK, and let NO be the



evolute. Then (Art. 101. Ex. 1.) $CO = \frac{my}{n}$, which is to SC ($=y$) as $m : n$, a constant ratio; and the angle SCO the complement of SCy being constant, the triangle SCO is given in specie; hence, SOC is constant, and CO being a tangent to the evolute NO, that curve is also a logarithmic spiral whose centre is S. Now $m : n :: y : v :: CO : SC$, but $y : v :: SC : CK$; therefore $CO : SC :: SC : CK$, hence, the triangle SCO is similar to SCK, and therefore the angle $SOK = SCy$; hence, the spirals are similar and equal.

The value of CO (corrected if necessary) gives the length of the evolute.

SECT. X.

ON LOGARITHMS, AND EXPONENTIAL QUANTITIES.

PROP. XLIX.

Given a number, to find it's logarithm.

(103.) LET $1 + x$ be the number, y it's logarithm, and m the modulus; then (Art. 44.) $\dot{y} = \frac{m\dot{x}}{1+x} = m \times (\dot{x} - x\dot{x} + x^2\dot{x} - x^3\dot{x} + \&c.)$ hence, by taking the fluents, $y = m \times (x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \&c.)$ which wants no correction, because when $x = 0$, y vanishes as it ought, for then the number becomes 1, whose log. $= 0$. Now this series will converge quicker the smaller x is. If $x = 1$, $y = m \times (1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \&c.) =$ the log. of 2. If $m = 1$, $y = 1 - \frac{1}{2} + \frac{1}{3} - \&c.$ the h. l. of 2. Hence, as we are at liberty to assume m what we please, we may, to the same number, have as many different systems of logarithms as we please.

If $m = 1$, and x be very small, the h. l. of $1 + x$ is x very nearly.

(104.) But to find a series which shall converge quicker, let the given number be $\frac{1+x}{1-x}$; then (Art. 44.)

$$\dot{y} = 2m \times \frac{\dot{x}}{1-x^2} = 2m \times (\dot{x} + x^2\dot{x} + x^4\dot{x} + \&c.)$$ whose fluent is $y = 2m \times (x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \&c.)$ If $m = 1$, we get $y = 2 \times (x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \&c.)$ for the hyp. log. of

$\frac{1+x}{1-x}$. Let $x = \frac{1}{3}$, and then the number becomes 2; hence,

$$\left.\begin{array}{rcr} x &=& 0{,}33333333\\ \frac{1}{3}x^3 &=& 1234567\\ \frac{1}{5}x^5 &=& 82307\\ \frac{1}{7}x^7 &=& 6532\\ \frac{1}{9}x^9 &=& 564\\ \frac{1}{11}x^{11} &=& 51\\ \hline && 0{,}34657354\\ && 2\\ \hline && 0{,}69314708\\ \hline \end{array}\right\}$$

This h. l. of 2 is true to 6 places; the true value to 7 places being 0,6931472; and it would have required at least 100000 terms of the series in Art. 103, to have given the value with the same degree of accuracy.

(105.) The common log. of 2 is 0,3010300. Now these different values depend on the different values of m, and in the former case $m = 1$; hence, 0,6931472 : 0,3010300 :: 1 : m in the latter case $=$,43429448 the modulus of the common system. Hence, if any common log. be *divided* by this modulus, it gives the corresponding hyp. log. Or if any hyp. log. be *multiplied* by it, it gives the corresponding common logarithm. For the various methods which have been invented to calculate logarithms, the reader is referred to Dr. HUTTON's very excellent Introduction to his Tables of Logarithms, and to Mr. MASERES's *Scriptores Logarithmici.*

(106.) By Art. 42. a set of quantities A^0, A^1, A^2, A^3, A^4, &c. in geometric progression will have their logarithms in arithmetic progression; hence, the indices 0, 1, 2, 3, 4, &c. may represent the respective logarithms. Now in the common system of logarithms, $A = 10$; hence, the logarithms of 10^0, 10^1, 10^2, 10^3, 10^4, &c. or of 1, 10, 100, 1000, 10000, &c. are 0, 1, 2, 3, 4, &c. And if between 10^0 and 10^1, we insert an indefinite number of geometric means, as 10^n, 10^{2n}, 10^{3n}, &c. n being indefinitely small, then some of these means must necessarily make up all the intermediate numbers between 1 and 10, as 2, 3, 4, 5, 6, 7, 8, 9, or at least be indefinitely near to them; the indices therefore of such means must be the logarithms of these numbers; for instance, if $10^{rn}=2$, then rn = log. of 2; if $10^{sn}=7$, then sn = log. of 7; and so for any other number.

DEFINITION.

(107.) The *measure of a ratio* 1 : *N* is the number of times which any other assumed ratio 1 : *A* must be taken to make that ratio. Thus, if $N = A^2$, the measure of the ratio of 1 : A^2 is 2, that ratio containing 2 ratios of 1 : *A*.

(108.) The ratio of 1 : A^2, 1 : A^3, 1 : A^4, &c. contain 2, 3, 4, &c. ratios of 1 : *A*; hence, the indices of *A* express the number of ratios of 1 : *A* which that ratio contains; for instance, 1 : A^4 contains 4 ratios of 1 : *A*; hence, 4 is the measure of the ratio 1 : A^4; also, the measure of the ratio of 1 : A^m is m, that ratio containing m ratios of 1 : *A*. Now if we put $A = 10$, then the measure of the ratio of 1 : 10^m is m; but by Article 106, m is the logarithm of 10^m; hence, the logarithm of any number is the measure of the ratio of that number to unity. In this sense, logarithms are called the measures of ratios, the logarithm of any number *N* showing how many ratios of 1 : 10 are necessary to make the ratio of 1 : *N*.

Hence, every ratio 1 : *N* has some certain measure in every system; now that ratio whose measure is m, the modulus of the system, is called the *Modular Ratio* by Mr. Cotes.

(109.) If $x=y^n$, then by taking the logarithms of both sides (*Trig*. Art. 6.), log. $x=n\times$ log. y; hence, if we have any equation of this form, log. $x=n\times$ log. y, then will $x=y^n$. If y be constant and n variable, the curve denoted by this equation is called the *logarithmic* curve. Further, if $\frac{m\dot{x}}{x} = \frac{\dot{y}}{y}$, then $m \times$ h. l. $x =$ h. l. y, and $x^m = y$, or $x^m = ay$, for this gives back the given fluxional equation.

LEMMA.

(110.) If $\left\{\begin{matrix} A+Bx+Cx^2+Dx^3+\&c. \\ a+bx+cx^2+dx^3+\&c. \end{matrix}\right\}=0$, or $(A+a)+(B+b)\times x+(C+c)\times x^2+(D+d)\times x^3+\&c.=0$, whatever be the value of x; then must $A+a=0$, $B+b=0$, $C+c=0$, &c. For as we may take x of any value, let $x=0$, and then $A+a=0$; hence, the remaining part, $(B+b)\times x+(C+c)\times x^2+(D+d)\times x^3+\&c.=0$, and dividing by x, $(B+b)+(C+c)\times x+(D+d)\times x^2+\&c.=0$; let $x=0$, and then $B+b=0$; and thus we may proceed for all the coefficients. Or we may consider it thus: The equation cannot become $=0$, but when it's roots are substituted for x; the equation therefore cannot vanish for *every* value of x you may assume, unless you make each term vanish, independent of x, by making each coefficient $=0$.

PROP. L.

Given a logarithm, to find it's number.

(111.) Let $1+x$ be any number and y it's logarithm, then $\dot{y}=\frac{m\dot{x}}{1+x}$; hence, $\dot{y}+x\dot{y}=m\dot{x}$, and $\dot{y}+x\dot{y}-m\dot{x}=0$. Assume $x=ay+by^2+cy^3+\&c.$ then $\dot{x}=a\dot{y}+2by\dot{y}+3cy^2\dot{y}+\&c.$ substitute these values of x and $\dot{x}$ into $\dot{y}+x\dot{y}-m\dot{x}=0$, and we have,

$$\left.\begin{matrix} \dot{y}+ & ay\dot{y}+ & by^2\dot{y}+\&c. \\ -ma\dot{y}- & 2mby\dot{y}- & 3mcy^2\dot{y}-\&c. \end{matrix}\right\}=0;$$

hence, (Art. 110.) $1-ma=0$, $a-2mb=0$, $b-3mc=0$, &c. therefore $a=\frac{1}{m}$; $b=\frac{a}{2m}=\frac{1}{2m^2}$; $c=\frac{b}{3m}=\frac{1}{2.3m^3}$; &c. hence, $x=\frac{y}{m}+\frac{y^2}{2m^2}+\frac{y^3}{2.3m^3}+\&c.$ consequently

$1+x=1+\frac{y}{m}+\frac{y^2}{2m^2}+\frac{y^3}{2.3m^3}+$ &c. the number whose logarithm is y.

If $m=1$, then $1+x=1+y+\frac{y^2}{2}+\frac{y^3}{2.3}+$ &c. is the number whose h. l. is y.

COR. If a be that number whose h. l. $=1$, then the h. l. of a^y is $y\times$ h. l. $a=y$. Hence, $a^y=1+y+\frac{y^2}{2}+\frac{y^3}{2.3}+$ &c.; if y be negative, $a^{-y}=1-y+\frac{y^2}{2}-\frac{y^3}{2.3}+$&c. Hence,

$$\frac{a^y+a^{-y}}{2}=1+\frac{y^2}{2}+\frac{y^4}{2.3.4}+\text{\&c.}$$

$$\frac{a^y-a^{-y}}{2}=y+\frac{y^3}{2.3}+\frac{y^5}{2.3.4.5}+\text{\&c.}$$

PROP. LI.

To find the modular ratio.

(112.) By Art. 108. every logarithm is the measure of the ratio of it's corresponding number to 1; hence, y is the measure of the ratio of $1+\frac{y}{m}+\frac{y^2}{2m^2}+\frac{y^3}{2.3m^3}$ +&c. to 1; now (Art. 108.) the modular ratio is that ratio of which the modulus is the measure; hence, if we make $m=y$, m will become the measure of the above ratio, and the ratio will become the modular ratio; making therefore $m=y$, the ratio becomes $1+1+\frac{1}{2}+\frac{1}{2.3}+$&c. $=2{,}7182818$ to 1, the modular ratio, which is therefore the same for every system, it being independent both of m and y.

(113.) A quantity is called an *exponential*, when it's index is variable.

PROP. LII.

To find the fluxion of the exponential x^y.

(114.) Put $x^y = z$, and let $X =$ h. l. x, $Z =$ h. l. z; then by the nature of logarithms, $yX = Z$, therefore $y\dot{X} + X\dot{y} = \dot{Z}$; but by Art. 45. $\dot{X} = \frac{\dot{x}}{x}$, and $\dot{Z} = \frac{\dot{z}}{z}$; hence, $\frac{y\dot{x}}{x} + X\dot{y} = \frac{\dot{z}}{z}$, consequently $\dot{z} = \frac{zy\dot{x}}{x} + zX\dot{y} = yx^{y-1}\dot{x} + Xx^y\dot{y}$.

If x be constant, then $\dot{x} = 0$, and $\dot{z} = Xx^y\dot{y}$.

If y be constant, $\dot{y} = 0$, and $\dot{z} = yx^{y-1}\dot{x}$, as in Art. 11.

PROP. LIII.

To find the fluxion of the exponential x^{y^z}.

(115.) Put $z^{y^z} = w$, and let $x^y = v$, then $v^z = w$; hence, if $V =$ h. l. v, we have (Art. 114.) $\dot{w} = zv^{z-1}\dot{v} + Vv^z\dot{z}$; but $v = x^y$, and $\dot{v} = yz^{y-1}\dot{x} + Xx^y\dot{y}$; hence, by substitution, $\dot{w} = zx^{y^{z-1}} \times (yx^{y-1}\dot{x} + Xx^y\dot{y}) + Vx^{y^z}\dot{z} = zyx^{y^{z-1}} \times x^{y-1}\dot{x} + zXx^{y^{z-1}} \times x^y\dot{y} + Vx^{y^z}\dot{z}$. If any one of the quantities x, y, z, become constant, it's fluxion $= 0$, and the term vanishes where that fluxion enters. In like manner we may find the fluxion, whatever be the number of quantities. The meaning of this notation is, the z power of x^y, not the y^z power of x. If this latter had been the meaning of the notation, we must have put $y^z = v$, instead of $x^y = v$.

SECT. XI.

ON THE FLUENTS OF QUANTITIES.

PROP. LIV.

Let $\dot{F} = \frac{z^{\frac{1}{2}n-1}\dot{z}}{a^n + z^n}$, *to find* F.

(116.) PUT $a^n = b^2$, $z^n = x^2$, then $z^{\frac{1}{2}n} = x$, $\therefore \frac{n}{2} \times z^{\frac{1}{2}n-1}\dot{z} = \dot{x}$, and $z^{\frac{1}{2}n-1}\dot{z} = \frac{2}{n} \times \dot{x}$; hence, $\dot{F} = \frac{2}{n} \times \frac{\dot{x}}{b^2 + x^2} = \frac{2}{n\,b^2} \times \frac{b^2\dot{x}}{b^2 + x^2}$; consequently (Art. 46.) $F = \frac{2}{n\,b^2} \times$ circ. arc, whose rad. $= b$, tan. $= x$.

PROP. LV.

Let $\dot{F} = \frac{z^{\frac{1}{2}n-1}\dot{z}}{a^n - z^n}$, *to find* F.

(117.) By the same substitution, $\dot{F} = \frac{2}{n} \times \frac{\dot{x}}{b^2 - x^2} = \frac{1}{n\,b} \times \frac{2\,b\,\dot{x}}{b^2 - x^2}$; hence, (Art. 45.) $F = \frac{1}{n\,b} \times$ h. l. $\frac{b+x}{b-x}$.

PROP. LVI.

Let $\dot{F} = \frac{z^{\frac{1}{2}n-1}\dot{z}}{\sqrt{a^n + z^n}}$, *to find* F.

(118.) By the same substitution, $\dot{F} = \frac{2}{n} \times \frac{\dot{x}}{\sqrt{b^2+x^2}}$; hence, (Art. 45.) $F = \frac{2}{n} \times \text{h. l.} \ (x + \sqrt{b^2+x^2})$.

Prop. LVII.

Let $\dot{\mathrm{F}} = \frac{\mathrm{z}^{\frac{1}{2}n-1}\dot{\mathrm{z}}}{\sqrt{\mathrm{a}^n - \mathrm{z}^n}}$, *to find* F.

(119.) By the same substitution, $\dot{F} = \frac{2}{n} \times \frac{\dot{x}}{\sqrt{b^2-x^2}}$ $= \frac{2}{nb} \times \frac{b\dot{x}}{\sqrt{b^2-x^2}}$; hence, (Art. 46.) $F = \frac{2}{nb} \times$ cir. arc, rad. $=b$, sine $=x$.

Prop. LVIII.

Let $\dot{\mathrm{F}} = \frac{\dot{\mathrm{z}}}{\sqrt{\mathrm{a}\mathrm{z}^2 + \mathrm{b}\mathrm{z} + \mathrm{c}}}$, *to find* F.

(120.) $\dot{F} = \frac{1}{\sqrt{a}} \times \frac{\dot{z}}{\sqrt{z^2 + \frac{b}{a} \times z + \frac{c}{a}}}$; put $z + \frac{b}{2a} = x$, then $z^2 + \frac{b}{a}z + \frac{b^2}{4a^2} = x^2$; hence, $z^2 + \frac{b}{a}z + \frac{c}{a} = x^2 - \frac{b^2}{4a^2}$ $+ \frac{c}{a} =$ (by putting $\frac{c}{a} - \frac{b^2}{4a^2} = d^2$) $x^2 + d^2$; also, $\dot{z} = \dot{x}$; hence, $\dot{F} = \frac{1}{\sqrt{a}} \times \frac{\dot{x}}{\sqrt{x^2+d^2}}$; and (Art. 45.) $F = \frac{1}{\sqrt{a}}$ $\times \text{h. l.} \ (x + \sqrt{x^2+d^2})$.

Prop. LIX.

Let $\dot{\mathrm{F}} = \frac{\mathrm{x}^{n-1}\dot{\mathrm{x}}}{\sqrt{\mathrm{a}\mathrm{x}^{2n} + \mathrm{b}\mathrm{x}^n + \mathrm{c}}}$, *to find* F.

(121.) Put $x^n = z$, then $x^{n-1}\dot{x} = \frac{1}{n} \times \dot{z}$; also, $x^{2n} = z^2$;

hence, $\dot{F} = \frac{1}{n} \times \frac{\dot{z}}{\sqrt{az^2+bz+c}}$, whose fluent is given in the last Proposition.

Prop. LX.

Let $\dot{F} = \frac{z^r \dot{z}}{\sqrt{az^2+bz+c}}$, *to find* F.

(122.) Let $x = \frac{b}{2a} + z$, then $z^2 + \frac{b}{a}z + \frac{c}{a}$ = (by Prop. 58.) $x^2 + d^2$; also, $z^{r+1} = \overline{x - \frac{b}{2a}}\Big|^{r+1}$, and $z^r \dot{z} = \overline{x - \frac{b}{2a}}\Big|^{r} \times \dot{x}$;

hence, $\dot{F} = \frac{1}{\sqrt{a}} \times \frac{\overline{x - \frac{b}{2a}}\Big|^{r} \times \dot{x}}{\sqrt{x^2+d^2}}$; expand the numerator, and taking the terms separately, the fluents of those terms where the index of x' in the numerator is *odd* are found by Art. 41; and where they are *even* by Art. 127.

Prop. LXI.

Let $\dot{F} = \frac{x^{rn-1}\dot{x}}{\sqrt{ax^{2n}+bx^n+c}}$, *to find* F.

(123.) Put $x^n = y$, then $x^{rn} = y^r$, and $x^{rn-1}\dot{x} = \frac{y^{r-1}\dot{y}}{n}$; hence, $\dot{F} = \frac{1}{n} \times \frac{y^{r-1}\dot{y}}{\sqrt{ay^2+by+c}}$ whose fluent is found by Prop. 59.

Prop. LXII.

Let $\dot{F} = \frac{x^2\dot{x}}{\sqrt{a^2+x^2}}$, *to find* F.

(124.) Assume $\dot{v} = \frac{\dot{x}}{\sqrt{a^2+x^2}}$, then (Art. 45.) $v =$ h. l.

$(x+\sqrt{a^2+x^2})$; put $w=\sqrt{a^2x^2+x^4}$, then $\dot{w}=\dfrac{a^2x\dot{x}+2x^3\dot{x}}{\sqrt{a^2x^2+x^4}}=\dfrac{a^2\dot{x}}{\sqrt{a^2+x^2}}+\dfrac{2x^2\dot{x}}{\sqrt{a^2+x^2}}=a^2\dot{v}+2\dot{F}$; hence, $\dot{F}=\frac{1}{2}\dot{w}-\frac{1}{2}a^2\dot{v}$, and $F=\frac{1}{2}w-\frac{1}{2}a^2v$. Call this P.

Prop. LXIII.

Let $\dot{\mathrm{F}}=\dfrac{\mathrm{x}^2\dot{\mathrm{x}}}{\sqrt{\mathrm{a}^2-\mathrm{x}^2}}$, *to find* F.

(125.) Assume $\dot{v}=\dfrac{a\dot{x}}{\sqrt{a^2-x^2}}$, then (Art. 46.) $v=$ cir. arc, rad. $=a$, sin. $=x$; put $w=\sqrt{a^2x^2-x^4}$, then $\dot{w}=\dfrac{a^2x\dot{x}-2x^3\dot{x}}{\sqrt{a^2x^2-x^4}}=\dfrac{a^2\dot{x}}{\sqrt{a^2-x^2}}-\dfrac{2x^2\dot{x}}{\sqrt{a^2-x^2}}=a\dot{v}-2\dot{F}$; hence, $\dot{F}=\frac{1}{2}a\dot{v}-\frac{1}{2}\dot{w}$, and $F=\frac{1}{2}av-\frac{1}{2}w$. Call this Q.

Cor. Hence we get the fluent of $\dfrac{\dot{x}}{x}\times\sqrt{2ax-x^2}$ $=x^{\frac{1}{2}-1}\dot{x}\sqrt{2a-x}$; for put $\sqrt{2a-x}=y$, and $x=2a-y^2$, $x^{\frac{1}{2}}=\overline{2a-y^2}|^{\frac{1}{2}}$, therefore $x^{\frac{1}{2}-1}\dot{x}=-\dfrac{y\dot{y}}{\overline{2a-y^2}|^{\frac{1}{2}}}$; hence the given fluxion becomes $\dfrac{-y^2\dot{y}}{\sqrt{2a-y^2}}$ the same form as above.

If $\dot{F}=\dfrac{x^2\dot{x}}{\sqrt{x^2-a^2}}$, put $\dot{v}=\dfrac{\dot{x}}{\sqrt{x^2-a^2}}$, and (Art. 45.) $v=\text{h. l.}\,(x+\sqrt{x^2-a^2})$. Assume $w=\sqrt{x^4-a^2x^2}$, then $\dot{w}=2\dot{F}-a^2\dot{v}$, and $F=\frac{1}{2}w+\frac{1}{2}a^2v$.

Prop. LXIV.

Let $\dot{\mathrm{F}}=\dfrac{\mathrm{x}^4\dot{\mathrm{x}}}{\sqrt{\mathrm{a}^2+\mathrm{x}^2}}$, *to find* F.

(126.) Assume $v = \sqrt{a^2x^6 + x^8}$, then $\dot{v} = \frac{3a^2x^5\dot{x} + 4x^7\dot{x}}{\sqrt{a^2x^6 + x^8}} = \frac{3a^2x^2\dot{x}}{\sqrt{a^2 + x^2}} + \frac{4x^4\dot{x}}{\sqrt{a^2 + x^2}} =$ (Art. 124.) $3a^2\dot{P} + 4\dot{F}$; hence, $\dot{F} = \frac{1}{4}\dot{v} - \frac{3a^2}{4}\dot{P}$, and $F = \frac{1}{4}v - \frac{3a^2}{4}P$.

PROP. LXV.

Let $\dot{F} = \frac{x^4\dot{x}}{\sqrt{a^2 - x^2}}$, *to find* F.

(127.) Assume $v = \sqrt{a^2x^6 - x^8}$, then $\dot{v} = \frac{3a^2x^5\dot{x} - 4x^7\dot{x}}{\sqrt{a^2x^6 - x^8}} = \frac{3a^2x^2\dot{x}}{\sqrt{a^2 - x^2}} - \frac{4x^4\dot{x}}{\sqrt{a^2 - x^2}} =$ (Art. 125.) $3a^2\dot{Q} - 4\dot{F}$; hence, $\dot{F} = \frac{3a^2}{4}\dot{Q} - \frac{1}{4}\dot{v}$, and $F = \frac{3a^2}{4}Q - \frac{1}{4}v$.

In this manner you may continue the fluents when the numerators are $x^6\dot{x}$, $x^8\dot{x}$, $x^{10}\dot{x}$, &c. by assuming $x = \sqrt{a^2x^{10} \pm x^{12}}$, $\sqrt{a^2x^{14} \pm x^{16}}$, $\sqrt{a^2x^{18} \pm x^{20}}$, &c. respectively, and by taking the fluxion, you will, in like manner, get $\dot{v}$ in terms of the given fluxion and of the next inferior fluxion.

PROP. LXVI.

Let $\dot{F} = x^n\dot{x}\sqrt{a^2 \pm x^2}$, n *being an even number, to find* F.

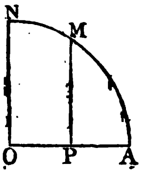
(128.) Multiply and divide the fluxion by $\sqrt{a^2 \pm x^2}$, and $\dot{F} = \frac{a^2x^n\dot{x} \pm x^{n+2}\dot{x}}{\sqrt{a^2 \pm x^2}}$; hence, as the indices of x in the numerator are even numbers, the fluents of $\frac{a^2x^n\dot{x}}{\sqrt{a^2\pm x^2}}$, and $\frac{x^{n+2}\dot{x}}{\sqrt{a^2\pm x^2}}$, may each be found by the method directed in the last Proposition.

If *n* be an odd number, *F* may be found by Art. 41.

Prop. LXVII.

Let $\dot{F} = \dot{x}\sqrt{2ax-x^2}$, *to find* F.

(129.) Let the radius $AO = a$, $AP = x$, then the sine $PM = \sqrt{2ax - x^2}$, therefore $\dot{F} = \dot{x}\sqrt{2ax - x^2} =$



(Art. 49.) the fluxion of the area *AMP*; hence, $F =$ the area *AMP*.

Prop. LXVIII.

Let $\dot{F} = x\dot{x}\sqrt{2ax-x^2}$, *to find* F.

(130.) Assume $w = \frac{1}{3}\times\overline{2ax-x^2}|^{\frac{3}{2}}$, then $\dot{w} = (a\dot{x} - x\dot{x}) \times \sqrt{2ax - x^2} = a\dot{x}\sqrt{2ax - x^2} - \dot{F}$; hence, $\dot{F} = a\dot{x}\sqrt{2ax - x^2} - \dot{w}$, and $F = a \times$ area $AMP - w$.

Prop. LXIX.

Let $\dot{F} = \frac{x\dot{x}}{\sqrt{2ax-x^2}}$, *to find* F.

(131.) Assume $w = \sqrt{2ax - x^2}$, then $\dot{w} = \frac{a\dot{x} - x\dot{x}}{\sqrt{2ax - x^2}}$

$$= \frac{a\dot{x}}{\sqrt{2ax - x^2}} - \frac{x\dot{x}}{\sqrt{2ax - x^2}} = \frac{a\dot{x}}{\sqrt{2ax - x^2}} - \dot{F};$$

hence, $\dot{F} = \frac{a\dot{x}}{\sqrt{2ax - x^2}} - \dot{w}$, and (Art. 46.) $F = z - w$, z being a circ. arc, rad. $= a$, versed sine $= x$.

PROP. LXX.

Let $\dot{F} = \frac{\dot{x}}{x}\sqrt{\frac{a+x}{x}}$, *to find* F.

(132.) Put $\sqrt{\frac{a+x}{x}} = y$, then $x = \frac{a}{y^2-1}$, and $\frac{\dot{x}}{x} = \frac{-2y\dot{y}}{y^2-1}$; hence, $\dot{F} = \frac{-2y^2\dot{y}}{y^2-1} = -2\dot{y} + \frac{2\dot{y}}{1-y^2}$, and $F = -2y + \text{h. l.}\,\frac{1+y}{1-y}$ (Art. 45. Ex. 6.). This fluent teaches us to find the length of the Cissoid of Diocles, the fluxion of the length being of this form.

PROP. LXXI.

Let $\dot{F} = \frac{x^m\dot{x}}{x-a}$, *to find* F.

(133.) Divide the num. by the den. till the index of x in the remainder $= 0$, and the remainder will then be $a^m\dot{x}$; hence, $\dot{F} = x^{m-1}\dot{x} + ax^{m-2}\dot{x} + a^2x^{m-3}\dot{x} + \&c. + a^m \times \frac{\dot{x}}{x-a}$; therefore (Art. 37. and 45.) $F = \frac{x^m}{m} + \frac{ax^{m-1}}{m-1} + \frac{a^2x^{m-2}}{m-2} + \&c. + a^m \times \text{h. l.}\,(x-a)$. Here m must be a whole positive number, otherwise the index of x cannot become $= 0$. If the denominator be $x+a$, the terms will be alternately $+$ and $-$.

If $m = \frac{1}{2}$, or $\dot{F} = \frac{x^{\frac{1}{2}}\dot{x}}{x-a}$, F may be thus found. Put $x = v^2$, then $x^{\frac{3}{2}} = v^3$, and $x^{\frac{1}{2}}\dot{x} = \frac{1}{2}v^2\dot{v}$; hence, $\dot{F} = \frac{\frac{1}{2}v^2\dot{v}}{v^2-a}$; but $\frac{1}{v^2-a} = \frac{1}{2a^{\frac{1}{2}}} \times \left(\frac{1}{v-a^{\frac{1}{2}}} - \frac{1}{v+a^{\frac{1}{2}}}\right) =$ if

$b=a^{\frac{1}{2}}$) $\frac{1}{2b}\times\left(\frac{1}{v-b}-\frac{1}{v+b}\right)$; therefore $\dot{F}=\frac{\frac{1}{2}v^2\dot{v}}{v^2-a}=$ $\frac{1}{2b}\times\left(\frac{v^2\dot{v}}{v-b}-\frac{v^2\dot{v}}{v+b}\right)$, and those fluents are found by his Proposition.

Prop. LXXII.

Let $\dot{\mathrm{F}}=\frac{\mathrm{z}^{\mathrm{rm}-1}\dot{\mathrm{z}}}{\mathrm{a}+\mathrm{b}\mathrm{z}^{\mathrm{m}}}$, *to find* F.

$$
\begin{array}{lll}
(134.)\ bz^m+a) & z^{rm-1}\dot{z} & \left(\frac{1}{b}\times z^{rm-m-1}\dot{z}-\frac{a}{b^2}\times z^{rm-2m-1}\dot{z}+\&c.\right. \\
 & z^{rm-1}\dot{z}+\frac{a}{b}\times z^{rm-m-1}\dot{z} & \\
\hline
 & -\frac{a}{b}\times z^{rm-m-1}\dot{z} & \\
 & -\frac{a}{b}\times z^{rm-m-1}\dot{z}-\frac{a^2}{b^2}\times z^{rm-2m-1}\dot{z} & \\
\hline
 & \frac{a^2}{b^2}\times z^{rm-2m-1}\dot{z} & \\
 & \&c.\ \&c. &
\end{array}
$$

continue this division till the index of z in the remainder becomes $m-1$, and the remainder will be $\pm\frac{a^{r-1}}{b^{r-1}}\times z^{m-1}\dot{z}$; hence, $\dot{F}=\frac{1}{b}\times z^{rm-m-1}\dot{z}-\frac{a}{b^2}\times z^{rm-2m-1}\dot{z}+\text{\&c.}\pm\frac{a^{r-1}}{b^{r-1}}\times\frac{z^{m-1}\dot{z}}{a+bz^m}$; now the last term $=\pm\frac{a^{r-1}}{mb^r}\times\frac{mbz^{m-1}\dot{z}}{a+bz^m}$; hence, (Art. 37. and 45.), $F=\frac{1}{b}\times\frac{z^{rm-m}}{rm-m}-\frac{a}{b^2}\times\frac{z^{rm-2m}}{rm-2m}+\text{\&c.}\pm\frac{a^{r-1}}{mb^r}\times$ h. l. $(a+bz^m)$. Here, r must be a whole positive number, otherwise the index of z can never become $m-1$.

If $\dot{F}=\frac{z^r\dot{z}}{1-z^2}=-z^{r-2}\dot{z}-z^{r-4}\dot{z}-\text{\&c.}+\frac{z\dot{z}}{1-z^2}$, or $\frac{\dot{z}}{1-z^2}$, according as r is an odd or even number, then $F=-\frac{z^{r-1}}{r-1}-\frac{z^{r-3}}{r-3}-\text{\&c.}-\frac{1}{2}$ h. l. $(1-z^2)$, or $+$ h. l. $\frac{1+z}{1-z}$.

If $\dot{F}=\frac{z^r\dot{z}}{1+z^2}$, the terms in the division go alternately $+$ and $-$, and the last term is $+$ or $-$, as the number of preceding terms is even or odd.

If $\dot{F}=\frac{z^{rm+\frac{1}{2}m-1}\dot{z}}{a+bz^m}$, the last term becomes $\pm\frac{a^r}{b^r}\times\frac{z^{\frac{1}{2}m-1}\dot{z}}{a+bz^m}$ whose fluent is found by Prop. 56.

PROP. LXXIII.

Let $\frac{1}{x^n-px^{n-1}+\text{\&c.}}=\frac{K}{x-a}+\frac{L}{x-b}+\frac{M}{x-c}+\text{\&c.}$ to find K, L, M, &c. where a, b, c, &c. are the roots of $x^n-px^{n-1}+\text{\&c.}=0$.

(135.) Reduce the fractions to a common denominator, and it will be the same as the denominator on the left, and consequently the sum of the numerators $=1$; hence $K \times (x-b) \times (x-c) \times \&c. + L \times (x-a) \times (x-c) \times \&c. + M \times (x-a) \times (x-b) \times \&c. = 1$; now as this is true let x be what it will, make $x=a$, and then $K \times (a-b) \times (a-c) \times \&c. = 1$, $\therefore K = \frac{1}{(a-b)\times(a-c)\times\&c.}$ Make $x=b$, and then $L \times (b-a) \times (b-c) \times \&c. = 1$, $\therefore$ $L = \frac{1}{(b-a)\times(b-c)\times\&c.}$. In like manner, we get the other numerators.

If $\frac{1}{(e+fz^m)\times(g+hz^m)} = \frac{K}{f\left(z^m+\frac{e}{f}\right)} + \frac{L}{h\left(z^m+\frac{g}{h}\right)}$, then in the same manner it appears, that $K = \frac{f}{fg-he}$ and $L = \frac{h}{he-fg}$.

PROP. LXXIV.

Let $\dot{F} = \frac{x^m \dot{x}}{x^n - px^{n-1} + \&c.}$, *to find* F, m *being a whole positive number.*

(136.) Let $\frac{1}{x^n - px^{n-1} + \&c.} = \frac{K}{x-a} + \frac{L}{x-b} + \&c.$ then K, L, &c. are known by the last Prop.; hence, $\frac{x^m \dot{x}}{x^n - px^{n-1} + \&c.} = \frac{Kx^m\dot{x}}{x-a} + \frac{Lx^m\dot{x}}{x-b} + \&c.$ Now (Art. 133.) the fluent of $\frac{Kx^m\dot{x}}{x-a}$ is $\frac{Kx^m}{m} + \frac{Kax^{m-1}}{m-1} + \&c. + Ka^m \times \text{h. l.}\,(x-a)$; in like manner, the fluents of all the other quantities are found, the sum of all which is F. Now the sum of all these quantities $= (K+L+\&c.)$

$\times \frac{x^m}{m} + (Ka + Lb + \&c.) \times \frac{x^{m-1}}{m-1} + \&c. + Ka^m \times$ h. l. $(x-a) + Lb^m \times$ h. l. $(x-b) + \&c.$ But by Dr. Waring's *Med. Alg. last edition in the Addenda*, $K + L + \&c. = 0$; $Ka + Lb + \&c. = 0$, &c. through all those terms, when m is less than n; in this case therefore $F = Ka^m \times$ h. l. $(x-a) + Lb^m \times$ h. l. $(x-b) + \&c.$ If m be equal to or greater than n, the coefficients of the first $n-1$ terms will become $= 0$.

(137.) If m be less than n, the quantity $\frac{x^m \dot{x}}{x^n - px^{n-1} + \&c.}$ may be resolved into $\frac{\dot{K}\dot{x}}{x-a} + \frac{\dot{L}\dot{x}}{x-b} + \frac{\dot{M}\dot{x}}{x-c} + \&c.$ for in this case $\dot{K} \times (x-b) \times (x-c) \times \&c. + \dot{L} \times (x-a) \times (x-c) \times \&c. + \&c. = x^m$; hence, if $x = a$, $\dot{K} = \frac{a^m}{(a-b) \times (a-c) \times \&c.}$; if $x = b$, $\dot{L} = \frac{b^m}{(b-a) \times (b-c) \times \&c.}$, &c. The reason why m must be less than n is this: The quantity $\dot{K} \times (x-b) \times (x-c) \times \&c. + \dot{L} \times (x-a) \times (x-c) \times \&c. + \&c. - x^m = 0$; and that this may be *always* true, the coefficients of the like powers of x must be assumed $= 0$ (Art. 110.), and by such an assumption you would deduce the same values of K, $\dot{L}$, &c. as above. Now the product of each of the quantities into which $\dot{K}$, $\dot{L}$, &c. are multiplied, is of $n-1$ dimensions in terms of x, there being $n-1$ factors; hence, if m be greater than $n-1$, there is only *one* term in which x is of m dimensions, therefore this term can never be made to vanish, generally with the rest. But if m be equal to or less than $n-1$, then this term x^m will come in with others having the same power, and the whole coefficient may be made $= 0$.

But the denominators may be otherwise expressed;

for as $(x-a)\times(x-b)\times$ &c. $=x^n-px^{n-1}+$ &c. by taking the fluxion we have $\dot{x}\times(x-b)\times(x-c)\times$ &c. $+\dot{x}\times(x-a)\times(x-c)\times$ &c. $+$ &c. $=nx^{n-1}\dot{x}-(n-1).px^{n-2}\dot{x}+$ &c. hence, if $x=a$, we have $(a-b)\times(a-c)\times$ &c. $=na^{n-1}-(n-1).pa^{n-2}+$ &c. If $x=b$, then $(b-a)\times(b-c)\times$ &c. $=nb^{n-1}-(n-1).pb^{n-2}+$ &c. and so on for the rest; hence, take the fluxion of the given equation, omitting $\dot{x}$, and write a, b, c, &c. for x, and we get the denominators.

Hence, when m is less than n, the fluent of $\dfrac{x^m\dot{x}}{x^n-px^{n-1}+\text{\&c.}}$ is $\dot{K}\times$ h. l. $(x-a)+\dot{L}\times$ h. l. $(x-b)+$ &c. which agrees with the conclusion in Art. 136. because $\dot{K}=Ka^m$, $\dot{L}=Lb^m$, &c.

(138.) If two roots, a, b, be equal, one of the quantities must have a quadratic divisor $(x-a)^2$. For example: Let $\dfrac{1}{x^3-px^2+qx-r}=\dfrac{Lx+M}{(x-a)^2}+\dfrac{N}{x-c}$: then reducing the two quantities on the right to the same denominator, and making the numerators equal, we get $Lx^2-Lcx+Mx-Mc+Nx^2-2Nax+Na^2-1=0$; hence, (Art. 110.) making $L+N=0$, $M-Lc-2Na=0$, $-Mc+Na^2-1=0$, we have, $L=-N$, $M=\dfrac{Na^2-1}{c}$; consequently $\dfrac{Na^2-1}{c}+Nc-2Na=0$; therefore $N=\dfrac{1}{\overline{a-c}|^2}$; $L=\dfrac{1}{-\overline{c-a}|^2}$; $M=\dfrac{2a-c}{\overline{a-c}|^2}$. Hence, the fluent of $\dfrac{\dot{x}}{x^3-px^2+qx-r}$, or $\dfrac{Lx\dot{x}+M\dot{x}}{\overline{x-a}|^2}+\dfrac{N\dot{x}}{x-c}$ may be thus found. Put $x-a=z$, then $x=z+a$, and $\dot{x}=\dot{z}$; hence, $\dfrac{Lx\dot{x}+M\dot{x}}{\overline{x-a}|^2}=\dfrac{Lz\dot{z}+La\dot{z}+M\dot{z}}{z^2}=$ (if $La+M=b$) $\dfrac{L\dot{z}}{z}+\dfrac{b\dot{z}}{z^2}$, whose fluent (Art. 45. and 37.) is $L \times$ h. l. $z - \frac{b}{z} = L \times$ h. l. $(x-a) - \frac{b}{x-a}$; and the fluent of $\frac{N\dot{x}}{x-c}$ is $N \times$ h. l. $(x-c)$.

(139.) If two of the roots be impossible, those two binomial fractions must be incorporated into one. Thus, let $\frac{1}{x^3-px^2+qx-r} = \frac{L}{x-a} + \frac{M}{x-b} + \frac{N}{x-c}$, and suppose a and b to be impossible; then $\frac{L}{x-a} + \frac{M}{x-b} = \frac{(L+M) \times x - (Lb+Ma)}{x^2-(a+b) \times x+ab}$, and the impossible quantities vanish, as will appear by substituting $m+n\sqrt{-1}$ for a, and $m-n\sqrt{-1}$ for b.

Prop. LXXV.

Let $\dot{F} = \frac{cx\dot{x}+d\dot{x}}{x^2-px+q}$, *to find* F.

(140.) Put $x-\frac{1}{2}p = z$, then $x = z+\frac{1}{2}p$, and $\dot{x}=\dot{z}$; hence, $d\dot{x}=d\dot{z}$, and $cx\dot{x}=cz\dot{z}+\frac{1}{2}pc\dot{z}$, $\therefore cx\dot{x}+d\dot{x} = cz\dot{z}+(\frac{1}{2}pc+d) \times \dot{z} =$ (if $\frac{1}{2}pc+d=e$) $cz\dot{z}+e\dot{z}$; also, $x^2-px+\frac{1}{4}p^2=z^2$; hence, $x^2-px+q=z^2+q-\frac{1}{4}p^2=$ (if $q-\frac{1}{4}p^2=a^2$) $z^2 \pm a^2$, according as a^2 is positive or negative, or according as the two values of x are impossible or possible. Hence, $\dot{F}=\frac{cz\dot{z}+e\dot{z}}{z^2 \pm a^2} = \frac{cz\dot{z}}{z^2 \pm a^2} + \frac{e\dot{z}}{z^2 \pm a^2}$. Now (Art. 45.) the fluent of $\frac{cz\dot{z}}{z^2 \pm a^2}$ is $\frac{1}{2}c \times$ h. l. $(z^2 \pm a^2)$. Also, taking $+a^2$, $\frac{e\dot{z}}{z^2+a^2} = \frac{e}{a^2} \times \frac{a^2\dot{z}}{z^2+a^2}$, whose fluent (Art. 46.) is $\frac{e}{a^2} \times$ cir. arc, rad.

$= a$, tan. $= z$. But taking $- a^2$, $\frac{e\dot{z}}{z^2 - a^2} = \frac{e}{2a} \times \frac{2a\dot{z}}{z^2 - a^2}$, whose fluent (Art. 45.) is $\frac{e}{2a} \times$ h. l. $\frac{z-a}{z+a}$; call the fluent of this second part B, and $F = \frac{1}{2}c \times$ h. l. $(z^2 \pm a^2) + B$. Call this fluent Q.

PROP. LXXVI.

Let $\dot{F} = \frac{x^m\dot{x}}{x^2 - px + q}$, *to find* F.

(141.) If the roots of $x^2 - px + q = 0$ be both possible, then (Art. 135.) resolve $\frac{1}{x^2 - px + q}$ into $\frac{K}{x-a} + \frac{L}{x-b}$; and $\dot{F} = \frac{Kx^m\dot{x}}{x-a} + \frac{Lx^m\dot{x}}{x-b}$, whose fluents are found by Art. 136. But if the roots be impossible, divide $x^m\dot{x}$ by $x^2 - px + q$ until the remainder becomes $cx\dot{x} + d\dot{x}$, c and d being put for the coefficients which arise from the division, and let the quotient be $x^{m-2}\dot{x} + ax^{m-3}\dot{x} + bx^{m-4}\dot{x} +$ &c. where $a = p$, $b = p^2 - q$, &c.; hence, $\dot{F} = x^{m-2}\dot{x} + ax^{m-3}\dot{x} + bx^{m-4}\dot{x} +$ &c. $+ \frac{cx\dot{x} + d\dot{x}}{x^2 - px + q}$, consequently (Art. 37. and 140.) $F = \frac{x^{m-1}}{m-1} + \frac{ax^{m-2}}{m-2} + \frac{bx^{m-3}}{m-3}$ + &c. $+ Q$.

If $m = 2$, then $F = x + Q$.
If $m = 3$, then $F = \frac{1}{2}x^2 + ax + Q$.
If $m = 4$, then $F = \frac{1}{3}x^3 + \frac{1}{2}ax^2 + bx + Q$.

PROP. LXXVII.

Let $\dot{F} = \frac{z^{-m}\dot{z}}{z^2 - pz + q}$, *to find* F.

(142.) Put $x = \frac{1}{z} = z^{-1}$, then $x^{m-1} = z^{-m+1}$, and $x^{m-2}\dot{x}$

$= -z^{-m}\dot{z}$; hence, $\dot{F} = \dfrac{-x^{m-2}\dot{x}}{\dfrac{1}{x^2} - \dfrac{p}{x} + q} = -\dfrac{x^m\dot{x}}{1 - px + qx^2}$

$= -\dfrac{1}{q} \times \dfrac{x^m\dot{x}}{\dfrac{1}{q} - \dfrac{px}{q} + x^2} = \left(\text{if } \dfrac{1}{q} = q', \dfrac{p}{q} = p'\right) - \dfrac{1}{q} \times$

$\dfrac{x^m\dot{x}}{x^2 - p'x + q'}$, which is the same as the last form.

Prop. LXXVIII.

Let $\dot{F} = \dfrac{\dot{z}}{z\sqrt{a + cz^n}}$, *to find* F.

(143.) First, $\dot{F} = \dfrac{1}{\sqrt{c}} \times \dfrac{\dot{z}}{z\sqrt{d^2 + z^n}}\left(\text{putting } d^2 = \dfrac{a}{c}\right)$; put $z^{\frac{1}{2}n} = x$, and then $z^n = x^2$; also, $\dfrac{\dot{x}}{x} = \dfrac{\frac{1}{2}nz^{\frac{1}{2}n-1}\dot{z}}{z^{\frac{1}{2}n}} = \frac{1}{2}n \times \dfrac{\dot{z}}{z}$, $\therefore \dfrac{2}{n} \times \dfrac{\dot{x}}{x} = \dfrac{\dot{z}}{z}$; hence, $\dot{F} = \dfrac{2}{n\sqrt{c}} \times \dfrac{\dot{x}}{x\sqrt{d^2 + x^2}}$ $= \dfrac{1}{nd\sqrt{c}} \times \dfrac{2d\dot{x}}{x\sqrt{d^2 + x^2}}$, and (Art. 45.) $F = \dfrac{1}{nd\sqrt{c}}$ $\times$ h. l. $\dfrac{\sqrt{d^2 + x^2} - d}{\sqrt{d^2 + x^2} + d}$. If d^2 be negative, $\dot{F} = \dfrac{2}{n\sqrt{c}}$ $\times \dfrac{\dot{x}}{x\sqrt{x^2 - d^2}} = \dfrac{2}{nd^2\sqrt{c}} \times \dfrac{d^2\dot{x}}{x\sqrt{x^2 - d^2}}$, and (Art. 46.) $F = \dfrac{2}{nd^2\sqrt{c}} \times$ cir. arc, rad. $= d$, secant $= x$.

Prop. LXXIX.

Let $\dot{F} = \dfrac{z\dot{z}\sqrt{b^2 + z^2}}{\sqrt{c^2 - z^2}}$, *to find* F.

(144.) Put $x = \sqrt{c^2 - z^2}$, then $z^2 = c^2 - x^2$, therefore

$z\dot{z} = -x\dot{x}$, and $\sqrt{b^2+z^2} = \sqrt{b^2+c^2-x^2} =$ (if $a^2 = b^2 + c^2$) $\sqrt{a^2-x^2}$; hence, $\dot{F} = -\dot{x}\sqrt{a^2-x^2}$. Now let *AN* be a circular arc whose centre is *O*, (See Fig. p. 167.) and *PM* be perpendicular to *AO*, and put $a = OA$, $x = OP$, then $PM = \sqrt{a^2-x^2}$; hence, $\dot{F} = -$the fluxion of the area *OPMN* (Art. 49.), consequently $F = -$ area *OPMN*.

Prop. LXXX.

Let $\dot{F} = \dfrac{z^{n-1}\dot{z}}{(g+hz^n)\sqrt{e+fz^n}}$, *to find* F.

(145.) Put $\sqrt{e+fz^n} = x$, then $z^n = \frac{x^2-e}{f}$, and $g + hz^n = g + \frac{h}{f}\times(x^2-e) = \frac{fg-eh}{f} + \frac{h}{f}x^2 = \left(\text{if } \frac{fg-eh}{f} = a, \frac{h}{f} = b\right) a + bx^2$; also, $nz^{n-1}\dot{z} = 2b\times x\dot{x}$, and $z^{n-1}\dot{z} = \frac{2b}{n}\times x\dot{x}$; hence, $\dot{F} = \frac{2b\dot{x}}{n\times(a+bx^2)}$, whose fluent is found by Art. 45. or 46, according as a and b have different or the same signs.

Prop. LXXXI.

Let $\dot{F} = \sqrt{\dfrac{e+fz^n}{g+hz^n}} \times x^{n-1}\dot{z}$, *to find* F.

(146.) Put $\sqrt{g+hz^n} = x$, then $z^n = \frac{x^2-g}{h}$, and $e + fz^n = e + \frac{f}{h}\times(x^2-g) = \frac{he-fg}{h} + \frac{f}{h}\times x^2 = \left(\text{if } \frac{he-fg}{h} = a, \frac{f}{h} = b\right) a + bx^2$; also, $z^{n-1}\dot{z} = \frac{2b}{n}\times x\dot{x}$; hence, $\dot{F} = \frac{2b}{n}\times\sqrt{a+bx^2}\times\dot{x}$, whose fluent is found by Art. 46.

when b is negative and a positive; but by Art. 45. when b is positive and a either positive or negative.

PROP. LXXXII.

Let $\dot{F} = \frac{z^{rm-1}\dot{z}}{(e+fz^m)\times(g+hz^m)}$ *to find* F, r *and* m *being whole positive numbers.*

(147.) By Art. 135. $\frac{z^{rm-1}\dot{z}}{(e+fz^m)\times(g+hz^m)} = \frac{Kz^{rm-1}\dot{z}}{e+fz^m} + \frac{Lz^{rm-1}\dot{z}}{g+hz^m}$, where K and L are known; and the fluents are found by Prop. 72.

If $\dot{F} = \frac{z^{rm+\frac{1}{2}m-1}\dot{z}}{(e+fz^m)\times(g+hz^m)}$, this resolves itself into $K\times\frac{z^{rm+\frac{1}{2}m-1}\dot{z}}{e+fz^m} + L\times\frac{z^{rm+\frac{1}{2}m-1}\dot{z}}{g+hz^m}$ (Art. 135.). And if the denominator of $\dot{F}$ be $e+fz^m+gz^{2m}$, this may be resolved into $(a+bz^m)\times(c+dz^m)$.

PROP. LXXXIII.

Let $\dot{F} = \overline{e+fz^n}\rvert^{m} \times \overline{g+hz^n}\rvert^{r} \times z^{sn-1}\dot{z}$, *to find* F, *where* s *is a whole positive number, and* r *half any whole positive number.*

(148.) Put $v = e+fz^n$, then $z^n = \frac{v-e}{f}$; $hz^n = \frac{h}{f}\times(v-e)$; $g+hz^n = g+\frac{h}{f}\times(v-e) = g-\frac{he}{f}+\frac{h}{f}\times v = \left(\text{if } d = g - \frac{he}{f}\right) d + \frac{h}{f}\times v$; $z^{sn} = \frac{1}{f^s}\times\overline{v-e}\rvert^{s}$; $snz^{sn-1}\dot{z} = \frac{s}{f^s}\times\overline{v-e}\rvert^{s-1}\dot{v}$; $z^{sn-1}\dot{z} = \frac{1}{nf^s}\times\overline{v-e}\rvert^{s-1}\dot{v}$; hence, by substitution we get $\dot{F} = v^m \times \overline{d+\frac{h}{f}v}|^{r} \times \frac{1}{nf^s} \times \overline{v-e}|^{s-1}\dot{v}$; and by expanding $\overline{d+\frac{h}{f}v}|^{r}$ and $\overline{v-e}|^{s-1}$, and actually multiplying each term into $v^m\dot{v}$, then when r is the half of an *odd* number (as $t+\frac{1}{2}$), $\overline{d+\frac{h}{f}v}|^{r} = \overline{d+\frac{h}{f}v}|^{t} \times \sqrt{d+\frac{h}{f}v}$, expand $\overline{d+\frac{h}{f}v}|^{t}$, and the fluent can be found by Art. 39. or 41. But when r is the half of an *even* number, expand $\overline{d+\frac{h}{f}v}|^{r}$, and then the fluent of each term may be found by Art. 37. except m be negative, such that one of the terms be of the form $\frac{\dot{v}}{v}$, in which case the fluent of that term is found by Art. 45.

If $r = -\frac{1}{2}$, and m a positive whole number, the fluent may be found by Art. 41. And if $m = -1$, then the fluent may be found by Art. 41. except for one term in the series thence arising, whose fluent is found by Prop. 78. it being of the form $\frac{\dot{v}}{v\sqrt{d+\frac{h}{f}v}}$.

PROP. LXXXIV.

Let $\dot{F} = \frac{\sqrt{a+bx^2}}{c+dx^2} \times \dot{x}$, *to find* F.

(149.) Multiply the num. and den. by $\sqrt{a+bx^2}$, and we get $\dot{F} = \frac{(a+bx^2)\times\dot{x}}{(c+dx^2)\times\sqrt{a+bx^2}} = \frac{a\dot{x}}{(c+dx^2)\times\sqrt{a+bx^2}}$

$+\frac{bx^2\dot{x}}{(c+dx^2)\times\sqrt{a+bx^2}}$. But the *first* of these terms $=\frac{ax^{-3}\dot{x}}{(d^2+cx^{-2})\times\sqrt{b+ax^{-2}}}$, and in the second term, by division $\frac{bx^2\dot{x}}{c+dx^2}=\frac{b}{d}\times\dot{x}-\frac{bc}{d}\times\frac{\dot{x}}{dx^2+c}$; hence, the *second* term $=\frac{b}{d}\times\frac{\dot{x}}{\sqrt{a+bx^2}}-\frac{bc}{d}\times\frac{\dot{x}}{(c+dx^2)\times\sqrt{a+bx^2}}$, and the last term of this $=-\frac{bc}{d}\times\frac{x^{-3}\dot{x}}{(d^2+cx^{-2})\times\sqrt{b+ax^{-2}}}$; hence, $\dot{F}=\frac{b}{d}\times\frac{\dot{x}}{\sqrt{a+bx^2}}+\left(a-\frac{bc}{d}\right)\times\frac{x^{-3}\dot{x}}{(d^2+cx^{-2})\times\sqrt{b+ax^{-2}}}$; now the fluent of the *first* of these terms is found by Art. 45, or 46, according to the signs of a and b, and of the *second* by Prop. 80.

LEMMA.

To resolve $\frac{1}{\overline{x+a}|^m\times\overline{x+b}|^n}$ into $\frac{H}{\overline{x+a}|^m}+\frac{K}{\overline{x+a}|^{m-1}}+\frac{L}{\overline{x+a}|^{m-2}}+\text{\&c.}+\frac{P}{\overline{x+b}|^n}+\frac{Q}{\overline{x+b}|^{n-1}}+\frac{R}{\overline{x+b}|^{n-2}}+$ &c. continued to m and n quantities respectively.

(150.) Reduce the fractions to a common denominator, and make the numerators on each side equal, and (A) $H\times(x+b)^n+K\times(x+b)^n\times(x+a)+L\times(x+b)^n\times(x+a)^2+\text{\&c.}+P\times(x+a)^m+Q\times(x+a)^m\times(x+b)+R\times(x+a)^m\times(x+b)^2+\text{\&c.}=1$. Make $x+a=0$, or $x=-a$, and every term where $x+a$ enters, becomes $=0$; hence, $H\times(x+b)^n=1$, or $H\times(b-a)^n=1$, $\therefore H=\frac{1}{(b-a)^n}$. Take the fluxion of the equation (*A*), and omitting $\dot{x}$, we have (*B*) $nH \times (x+b)^{n-1} + nK \times (x+b)^{n-1} \times (x+a) + K \times (x+b)^{n} + \&c. = 0$; make $x = -a$, and we have $nH \times (b-a)^{n-1} + K \times (b-a)^{n} = 0$; hence, $K = -\frac{nH}{b-a} = \frac{-n}{(b-a)^{n+1}}$; thus by continuing to take the fluxion of the last equation, and then making $x = -a$, we shall get the values of *L*, &c. In like manner, if we make $x + b = 0$, or $x = -b$, we find $P = \frac{1}{(a-b)^{m}}$; then by taking the fluxion of the last equation, and making $x = -b$, we get $Q = \frac{-m}{(a-b)^{m+1}}$; and by proceeding as before, we get *R*, &c.

Prop. LXXXV.

Let $\dot{F} = \frac{x^{r}\dot{x}}{(x+a)^{m} \times (x+b)^{n}}$, *to find* $\dot{F}$, r *being a whole positive number.*

(151.) By the last Lem., $\dot{F} = \frac{Hx^{r}\dot{x}}{(x+a)^{m}} + \frac{Kx^{r}\dot{x}}{(x+a)^{m-1}} + \&c. + \frac{Px^{r}\dot{x}}{(x+b)^{n}} + \frac{Qx^{r}\dot{x}}{(x+b)^{n-1}} + \&c.$ Put $x + a = z$, then $x = z - a$, therefore $x^{r+1} = (z-a)^{r+1}$, and $x^{r}\dot{x} = (z-a)^{r} \times \dot{z}$; hence, $\frac{x^{r}\dot{x}}{(x+a)^{m}} = \frac{(z-a)^{r} \times \dot{z}}{z^{m}} = z^{r-m}\dot{z} - raz^{r-m-1}\dot{z} + r.\frac{r-1}{2}a^{2}z^{r-m-2}\dot{z} - \&c.$ where the number of the terms $= r + 1$, and the fluent of every term is found by Art. 37. except that term where the index of z is -1, whose fluent is found by Art. 45. and the sum of all these multiplied by *H*, is the fluent of the first term. In like manner, the fluents of the other terms are found.

Prop. LXXXVI.

Let $\dot F = \frac{\dot x}{(a+bx)\times(c+dx)} \times \overline{\frac{a+bx}{c+dx}}\Big|^{\frac{m}{n}}$, *to find* F.

Put $z = \frac{a+bx}{c+dx}$, then $\frac{\dot z}{z} = \frac{(bc-ad)\times\dot x}{(a+bx)\times(c+dx)}$; hence,

$\dot F = \frac{z^{\frac{m}{n}-1}\dot z}{bc-ad}$, and $F = \frac{n z^{\frac{m}{n}}}{m\times(bc-ad)} = \frac{n}{m\times(bc-ad)} \times \overline{\frac{a+bx}{c+dx}}\Big|^{\frac{m}{n}}$.

Prop. LXXXVII.

Given A *the fluent of* $\overline{e+fx^n}|^m \times x^p\dot x$, *to find* B *the fluent of* $\overline{e+fx^n}|^m \times x^{p+n}\dot x$, *and* C *the fluent of* $\overline{e+fx^n}|^{m+1} \times x^p\dot x$.

(152.) Assume $Q = \overline{e+fx^n}|^{m+1}\times x^{p+1}$, then $\dot Q = (p+1)\times\overline{e+fx^n}|^{m+1}\times x^p\dot x + (m+1)\times nf\times\overline{e+fx^n}|^m\times x^{p+n}\dot x = (p+1)\times\dot C + (m+1)\times nf\times\dot B$; hence, by taking the fluents, $Q=(p+1)\times C+(m+1)\times nf\times B$. Also, $\overline{e+fx^n}|^{m+1}\times x^p\dot x = (e+fx)^n\times\overline{e+fx^n}|^m\times x^p\dot x = e\times\overline{e+fx^n}|^m\times x^p\dot x + f\times\overline{e+fx^n}|^m\times x^{p+n}\dot x$, that is, $\dot C = e\dot A + f\dot B$, therefore $C = eA + fB$. Now from the *first* fluent, $B = \frac{Q-(p+1)\times C}{(m+1)\times nf}$, and from the *second*, $B = \frac{C-eA}{f}$; hence, $\frac{Q-(p+1)\times C}{(m+1)\times nf} = \frac{C-eA}{f}$; ∴ $C = \frac{Q+(m+1)\times neA}{p+1+(m+1)\times n}$; consequently $B = \frac{C-eA}{f} =$

$\frac{C}{f} - \frac{eA}{f} = \frac{Q + (m+1) \times neA}{(p+1+(m+1) \times n) \times f} - \frac{eA}{f}$. Hence, we may continue the fluent as far as we please, increasing m by 1, and p by n.

Let $e = a^2$, $f = 1$, $m = -\frac{1}{2}$, $p = 0$, $n = 2$; then $\dot{A} = \frac{\dot{x}}{\sqrt{a^2 + x^2}}$, and $A = \text{h. l.}\ (x + \sqrt{x^2 + a^2})$ (Art. 45.); hence, B the fluent of $\frac{x^2 \dot{x}}{\sqrt{a^2 + x^2}} = \frac{1}{2} x \times \overline{a^2 + x^2}|^{\frac{1}{2}} - \frac{1}{2} a^2 A$, as in Art. 124. also, C the fluent of $\overline{a^2 + x^2}|^{\frac{1}{2}} \times \dot{x} = \frac{1}{2} x \times \overline{a^2 + x^2}|^{\frac{1}{2}} + \frac{1}{2} a^2 A$.

PROP. LXXXVIII.

Let $\mathrm{R} = \mathrm{e} + \mathrm{f}x^{n} + \mathrm{g}x^{2n} + \&\mathrm{c}.$, $\dot{\mathrm{A}} = \mathrm{R}^{s} x^{m-1} \dot{x}$, $\dot{\mathrm{B}} = x^{n} \dot{\mathrm{A}} = \mathrm{R}^{s} x^{m+n-1} \dot{x}$, $\dot{\mathrm{C}} = x^{n} \dot{\mathrm{B}} = \mathrm{R}^{s} x^{m+2n-1} \dot{x}$, $\dot{\mathrm{D}} = x^{n} \dot{\mathrm{C}} = \mathrm{R}^{s} x^{m+3n-1} \dot{x}$, &c. *given the fluents of all but one, to find that fluent.*

The fluxion of $x^m R^{s+1} = x^{m-1} R^{s} \left(mR\dot{x} + (s+1)x\dot{R}\right)$ = (substituting for R and $\dot{R}$ their values, and putting $p = sn + m$) $me\dot{A} + (p + m.f\dot{B} + (2p+m).g\dot{C} +$ &c. hence, $x^m R^{s+1} = meA + (p+m).fB + (2p+m)gC +$ &c. and if q be the number of terms in R, and $q - 1$ of these fluents be given, the other fluent is known, being the only unknown quantity in the equation.

PROP. LXXXIX.

Let $\mathrm{R} = \mathrm{e} + \mathrm{f}x^{n}$, *to find the fluent of* $x^{m+rn-1} \dot{x} \mathrm{R}^{s}$.

From the last Prop. we have $B = \frac{x^m R^{s+1}}{(p+m).f} - \frac{meA}{(p+m).f}$ the fluent of $x^{m+n-1} \dot{x} R^{s}$. Now as $\dot{B} =$

$R^s x^{m+n-1}\dot{x}$, to find C in terms of B, proceed in like manner, writing $m+n$ instead of m, and we get $C = \frac{x^{m+n}R^{s+1}}{(p+m+n).f} - \frac{(m+n).eB}{(p+m+n).f}$, the fluent of $R^s x^{m+2n-1}\dot{x}$. Also $\dot{C} = x^n\dot{B} = x^{m+2n-1}\dot{x}$; therefore to find D in terms of C, write $m+2n$ for m, and we get $D = \frac{x^{m+2n}R^{s+1}}{(p+m+2n).f} - \frac{(m+2n)\times eC}{(p+m+2n).f}$, the fluent of $x^{m+3n-1}\dot{x}R^s$. Here the law of continuation is manifest, and thus we go up to the fluent of $x^{m+rn-1}\dot{x}R^s$.

If $R=0$, $M=p+n$, then $B = -\frac{m}{M}\times\frac{eA}{f}$; $C = -\frac{m+n}{M+n}\times\frac{eB}{f} = \frac{m.(m+n)}{M\times M+n}\times\frac{e^2A}{f^2}$; $D = -\frac{m+2n}{M+2n}\times\frac{eC}{f} = -\frac{m.(m+n)\times(m+2)}{M.(M+n)\times(M+2n)}\times\frac{e^3C}{f^3}$; &c. and the fluent of $x^{m+rn-1}\dot{x}R^s$ generated whilst x flows from o to $-\overline{\frac{e}{f}}\Big|^{\frac{1}{n}} = \pm\frac{m.(m+n).(m+2n)\ldots\big(m+(r-1).n\big)}{M.(M+n).(M+2n\ldots\big(M+(r-1).n\big)}\times\frac{e^r}{f^r}\times A$, where the sign is + or − as r is even or odd.

If $R=e-fx^n$, and $R=0$, then the fluent generated whilst x flows from o to $\overline{\frac{e}{f}}\Big|^{\frac{1}{n}}$. is $\frac{m.(m+n)\ldots.m+(r-1).n}{M.(M+n)\ldots.\big(M+(r-1).n\big)}\times\frac{e^r}{f^r}\times\overline{\frac{e}{f}}\Big|^{\frac{1}{n}}$.

If $m=1$, $n=2$, $s=-\frac{1}{2}$, $e=1$, $f=1$, then $M=2$, and $\dot{A} = \frac{\dot{x}}{\sqrt{1-x^2}}$, $A=\frac{1}{4}$ circum. of a circle whose rad. $=1$; hence, the fluent of $\frac{x^{2r}\dot{x}}{\sqrt{1-x^2}}$, generated whilst x flows from o to 1, is $\frac{1\,.\,3\,.\,5\,...\,(2r-1)}{2\,.\,4\,.\,6\,....\,2r} \times A$, r being the number of factors.

If $\dot{A} = \dot{x}\sqrt{1-x^2}$, where A is the quadrantal area, the fluent of $x^r\dot{x}\sqrt{1-x^2}$, generated as above, is $\frac{1\,.\,3\,.\,5\,....\,(2r-1)}{2\,.\,4\,.\,6\,....\,2r} \times A$.

If $x = \frac{z}{\sqrt{1+z^2}}$, then $\frac{x^{2r}\dot{x}}{\sqrt{1-x^2}} = \frac{z^{2r}\dot{z}}{\overline{1+z^2}|^{r+1}}$, and the former fluent represents the fluent of this quantity generated whilst z flows from o to infinity, for when $x = 0$, $z = 0$, and when $x = 1$, z is infinite. In the second case, the fluxion becomes $\frac{z^{2r}\dot{x}}{\overline{1+z^2}|^{r+2}}$, and the latter fluent represents the fluent under the same limits.

If in the equation $x^m R^{s+1} = meA + (p+m)\,.\,fB$, neither m nor $p+m$ are equal to nothing, A and B mutually depend on each other. If $A = 0$, B is found in finite terms; if $p+m = 0$, A is found in finite terms.

As the series may be continued either way, let $R = fx^n - e$, $\dot{A} = x^{m-1}\dot{x}R^s$, $\dot{B} = x^{-n}\dot{A}$, $\dot{C} = x^{-n}\dot{B} = x^{-2n}\dot{A}\,.....\,\dot{Q} = x^{-rn}\dot{A}$, then we get $Q = \frac{(M-n).(M-2n)\,.....\,(M-rn)}{(m-n)\,.\,(m-2n)\,.....\,(m-rn)} \times \frac{f^r}{e^r} \times A$. If m be negative, so that $x^m R^{s+1}$ may become $\frac{R^{s+1}}{x^m}$, then this quantity becomes $= 0$ when x is infinite and when $x = \overline{\frac{e}{f}}\Big|^{\frac{1}{n}}$; hence the fluent here found is between these limits of x.

If $\dot{A} = \frac{\dot{x}}{x\sqrt{x^2-1}}$, then $\dot{Q} = \frac{\dot{x}}{x^{2r+1}\sqrt{1-x^2}}$, and $Q = \frac{1}{2} \times \frac{3}{4} \times \ldots \frac{2r-1}{2r} \times A$ within the above limits of x.

Sometimes the fluent may be found by transforming the fluxion from the sine or cosine of an arc to the tangent or secant; or the converse.

Prop. XC.

Let $\dot{\mathrm{F}} = \frac{\dot{\mathrm{x}}}{\mathrm{x}^2\sqrt{1-\mathrm{x}^2}}$, *to find* F.

To radius = 1, let x = sine, z = tangent of an arc; then (Art. 46.) $\frac{\dot{x}}{\sqrt{1-x^2}} = \frac{\dot{z}}{1+z^2}$; also, $x = \frac{z}{\sqrt{1+z^2}}$; hence, $\dot{F} = \frac{\dot{z}}{z^2}$, and $F = -\frac{1}{z} = -\frac{\sqrt{1-x^2}}{x}$.

Prop. XCI.

Let $\dot{\mathrm{F}} = \frac{\mathrm{a}^2\dot{\mathrm{x}}}{(\mathrm{a}^2 \mp \mathrm{x}^2) \times \sqrt{1-\mathrm{x}^2}}$, *to find* F.

By the same substitution as in the last Prop. we get $\dot{F} = \frac{a^2\dot{z}}{a^2 + (a^2 \mp 1) \times z^2} = \left(\text{if } b = \frac{a^2}{a^2 \mp 1}\right)\frac{b^2\dot{z}}{b^2 + z^2}$, whose fluent (Art. 46.) is F = a cir. arc, whose radius is b and tangent z, or $\frac{x}{\sqrt{1-x^2}}$.

Prop. XCII.

Let $\dot{\mathrm{F}} = \frac{\dot{\mathrm{x}}}{\mathrm{x}\sqrt{\mathrm{a}^2+\mathrm{x}^{2n}}}$, *to find* F.

The same notation remaining, assume $\frac{a+\sqrt{a^2+x^{2n}}}{x^n} = z$, then $\dot{F} = \frac{-1}{na} \times \frac{\dot{z}}{z}$; hence, $F = \frac{-1}{na} \times$ h. l. z.

PROP. XCIII.

Let $\dot{F} = v x^n \dot{x}$, *where* $v = h.\ l.\ \frac{1}{1-x}$, *to find* F.

(153.) Assume $\frac{vx^{n+1}}{n+1} + r = F$, then $vx^n\dot{x} + \frac{x^{n+1}\dot{v}}{n+1} + \dot{r} = \dot{F} = vx^n\dot{x}$; hence $\dot{r} = -\frac{x^{n+1}\dot{v}}{n+1} =$ (because $\dot{v} = \frac{\dot{x}}{1-x}$) $= -\frac{x^{n+1}\dot{x}}{(n+1)\times(1-x)} =$ (by division) $-\frac{1}{n+1} \times \left(-x^n\dot{x} - x^{n-1}\dot{x} - \&c. + \frac{\dot{x}}{1-x}\right)$, therefore $r = \frac{1}{n+1} \times \left(\frac{x^{n+1}}{n+1} + \frac{x^n}{n} + \&c. - v\right)$; hence, $F = \frac{vx^{n+1}}{n+1} + \frac{1}{n+1} \times \left(\frac{x^{n+1}}{n+1} + \frac{x^n}{n} + \&c. - v\right)$.

PROP. XCIV.

Let $\dot{F} = v x^n \dot{x}$, *where* v *is a circular arc whose radius is* 1 *and tangent* x, *to find* F.

(154.) Assume $\frac{vx^{n+1}}{n+1} + r = F$; then $vx^n\dot{x} + \frac{x^{n+1}\dot{v}}{n+1} + \dot{r} = \dot{F} = vx^n\dot{x}$. Let n be an *odd* number, and then $\dot{r} = -\frac{x^{n+1}\dot{v}}{n+1} =$ (Art. 46.) $-\frac{x^{n+1}\dot{x}}{(n+1)\times(1+x^2)} = -\frac{1}{n+1} \times (x^{n-1}\dot{x} - x^{n-3}\dot{x} + \&c. \pm \dot{v})$ $\left(\frac{\dot{x}}{1+x^2}\right)$, where the sign of

$\dot v$ will be + or − , according as $\frac{n+1}{2}$ is even or odd; hence, $r=\frac{1}{n+1}\times\left(-\frac{x^n}{n}+\frac{x^{n-2}}{n-2}-\&c.\mp v\right)$; therefore $F=\frac{vx^{n+1}}{n+1}+\frac{1}{n+1}\times\left(-\frac{x^n}{n}+\frac{x^{n-2}}{n-2}-\&c.\mp v\right)$. If n be an *even* number, the last term of the division will be $\pm\frac{x\dot x}{1+x^2}$, whose fluent is $\pm\frac{1}{2}$ h. l. $(1+x^2)$ = ± h. l. $\sqrt{1+x^2}$; hence, $F=\frac{vx^{n+1}}{n+1}+\frac{1}{n+1}\times\left(-\frac{x^n}{n}+\frac{x^{n-2}}{n-2}-\&c.\pm\text{h. l.}\sqrt{1+x^2}\right)$, where the sign of the last term is + or −, according as $\frac{1}{2}n$ is odd or even.

PROP. XCV.

Let $\dot F = z^m x^{n-1}\dot x$, *where* z = h. l. x, *to find* F.

(155.) Assume $F = az^m+bz^{m-1}+cz^{m-2}+\&c.$ a, b; c, &c. being variable coefficients in terms of x; hence, by taking the fluxion, we have,

$$\left.\begin{array}{llll}\dot a z^m + & \dot b z^{m-1} + & \dot c z^{m-2} & +\&c. \\ & m a \dot z z^{m-1} + & (m-1).b\dot z z^{m-2} & +\&c.\end{array}\right\} = z^m x^{n-1}\dot x;$$

but by Art. 45. $\dot z = \frac{\dot x}{x}$; hence, by transposition,

$$\left.\begin{array}{llll}\dot a z^m + & \dot b z^{m-1} + & \dot c z^{m-2} & +\&c. \\ -x^{n-1}\dot x z^m + & \frac{ma\dot x}{x}z^{m-1} + & \frac{(m-1).b\dot x}{x}z^{m-2} & +\&c.\end{array}\right\} = 0;$$

therefore, by Art. 110. $\dot a - x^{n-1}\dot x = 0$, $\dot b + \frac{ma\dot x}{x} = 0$, $\dot c + \frac{(m-1.)b\dot x}{x} = 0$, &c. hence, $\dot a = x^{n-1}\dot x$, ∴ $a = \frac{x^n}{n}$; $\dot b =$

$$\frac{-ma\dot{x}}{x} = \frac{-mx^{n-1}\dot{x}}{n}, \therefore b = \frac{-mx^n}{n^2};\ \dot{c} = \frac{-(m-1).b\dot{x}}{x}$$

$$= \frac{-(m-1)\times -mx^{n-1}\dot{x}}{n^2}, \therefore c = \frac{m.(m-1).x^n}{n^3};\ \text{\&c.}$$

hence, $F = \frac{x^n}{n} \times z^m - \frac{mx^n}{n^2} \times z^{m-1} + \frac{m.(m-1).x^n}{n^3} \times z^{m-2} - \text{\&c.}$ where the law of continuation is manifest, and the series will terminate when m is a whole positive number.

PROP. XCVI.

Let $\dot{F} = a^x x^n \dot{x}$, *to find* F.

(156.) Assume $F = a^x \times (px^n + qx^{n-1} + rx^{n-2} + \text{\&c.})$ and let $m = \text{h. l. } a$; then (Art. 114.) $ma^x\dot{x}$ is the fluxion of a^x; hence, by taking the fluxions,

$$\left.\begin{array}{l} ma^x\dot{x} \times (px^n + qx^{n-1} + \quad rx^{n-2} +)\ \text{\&c.} \\ a^x \ \times \ (npx^{n-1}\dot{x} + (n-1).qx^{n-2}\dot{x} +)\ \text{\&c.} \end{array}\right\} = a^x x^n \dot{x};$$

divide both sides by $a^x\dot{x}$, and transpose x^n, and we have

$$\left.\begin{array}{l} mpx^n + mqx^{n-1} + \quad mrx^{n-2} + \text{\&c.} \\ -x^n + npx^{n-1} + (n-1).qx^{n-2} + \text{\&c.} \end{array}\right\} = 0;$$

hence, (Art. 110.) $mp - 1 = 0$, $mq + np = 0$, $mr + (n-1).q = 0$, &c. $\therefore p = \frac{1}{m}$; $q = \frac{-np}{m} = -\frac{n}{m^2}$; $r = -\frac{(n-1).q}{m} = -\frac{(n-1)\times -n}{m^3} = \frac{n.(n-1)}{m^3}$ &c.; hence, $F = a^x \times \left(\frac{1}{m}x^n - \frac{n}{m^2}x^{n-1} + \frac{n.(n-1)}{m^3}x^{n-2} - \text{\&c.}\right)$ where the law of continuation is manifest, and the series will terminate when n is a whole number.

PROP. XCVII.

To find the fluent of $\frac{z^r\dot{z}}{\overline{1 \pm z^n}|^2}$, *given the fluent of* $\frac{z^r\dot{z}}{1 \pm z^n}$.

(157.) Assume $\frac{az^{r+1}}{1\pm z^n}+Q$ for the fluent; then, by taking the fluxion, we have $\frac{(r+1)\times az^r\dot z\times(1\pm z^n)\mp naz^{r+n}\dot z}{\overline{1\pm z^n}|^2}$ $+\dot Q=\frac{z^r\dot z}{\overline{1\pm z^n}|^2}$, or $\frac{z^r\dot z}{\overline{1\pm z^n}|^2}=$ $\frac{1}{1\pm z^n}\times\left((r+1)\times az^r\dot z\mp\frac{naz^{r+n}\dot z}{1\pm z^n}\right)+\dot Q$; but $\mp\frac{naz^{r+n}\dot z}{1\pm z^n}$ $=-naz^r\dot z+\frac{naz^r\dot z}{1\pm z^n}$; hence, $\frac{z^r\dot z}{\overline{1\pm z^n}|^2}=\frac{1}{1\pm z^n}\times$ $\left((r+1)\times az^r\dot z-naz^r\dot z+\frac{naz^r\dot z}{1\pm z^n}\right)+\dot Q=\left((r+1)\times a-na\right)\times$ $\frac{z^r\dot z}{1\pm z^n}+\frac{naz^r\dot z}{\overline{1\pm z^n}|^2}+\dot Q$; assume $na=1$, or $a=\frac{1}{n}$, so that the terms $\frac{z^r\dot z}{\overline{1\pm z^n}|^2}$ and $\frac{naz^r\dot z}{\overline{1\pm z^n}|^2}$ may destroy each other, and we have $\dot Q=\left(1-\frac{r+1}{n}\right)\times\frac{z^r\dot x}{1\pm z^n}$; hence, if P be the fluent of $\frac{z^r\dot z}{1\pm z^n}$, we have $Q=\left(1-\frac{r+1}{n}\right)\times P$; consequently the fluent required is $\frac{1}{n}\times\frac{z^{r+1}}{1\pm z^n}+\left(1-\frac{r+1}{n}\right)$ $\times P$.

Prop. XCVIII.

Let $\dot F=\frac{\dot x}{x}\sqrt{ax+x^2}$, *to find* F.

Put $\frac{\sqrt{ax+x^2}}{x}=y$, then $x=\frac{a}{y^2-1}$, $\dot x=\frac{-2ay\dot y}{\overline{1-y^2}|^2}$;

hence, $\dot{F} = \frac{-2ay^2\dot{y}}{\overline{1-y^2}|^2}$, and by the last Prop. $F = -2a\left(\frac{1}{2} \times \frac{y^3}{1-y^2} - \frac{1}{2}P\right)$.

Prop. XCIX.

$$Let\ \dot{\mathrm{F}} = \frac{\mathrm{a}\dot{\mathrm{x}} + \mathrm{b}\mathrm{x}\dot{\mathrm{x}}}{\mathrm{c}\mathrm{x} + \mathrm{x}^2},\ to\ find\ \mathrm{F}.$$

Assume $d \times$ h. l. $(cx^r + x^{r+1})$ for the fluent; then it's fluxion is $d \times \frac{rcx^{r-1}\dot{x} + (r+1)x^r\dot{x}}{cx^r + x^{r+1}} = \frac{dcr\dot{x} + d \times (r+1).x\dot{x}}{cx + x^2}$, which put $= \frac{a\dot{x} + bx\dot{x}}{cx + x^2}$, we get $dcr = a$, $d \times (r+1) = b$; therefore $r = \frac{a}{bc-a}$, $d = \frac{bc-a}{c}$; hence, $F = \frac{bc-a}{c} \times$ h. l. $\left(cx^{\frac{a}{bc-a}} + x^{\frac{bc}{bc-a}}\right)$.

Prop. C.

To find fluents where there are two variable quantities in the given fluxion.

(158.) It frequently happens, that a fluxional equation contains two variable quantities, in which case, they must either be separated, or reduced to the fluxion of some known fluent; but no general rules can be given for this purpose, and the reductions must be left to trial and the skill of the Analyst; the following Rules, however, may be of some use.

RULE. 1.

Multiply or divide the given equation by some function of the unknown quantities, so as to bring them to a form whose fluents may be found by some of the rules already given, or to the fluxion of a known fluent.

EXAMPLES.

Ex. 1. Let $\frac{\dot{x}}{x}+\frac{\dot{y}}{y}=\frac{ax^m\dot{x}}{y^n}$. Multiply both sides by nx^ny^n, and it becomes $ny^nx^{n-1}\dot{x}+nx^ny^{n-1}\dot{y}=nax^{m+n}\dot{x}$; now the fluent of the first part is known from Prop. 7. to be x^ny^n, and the fluent of the other part is found (Art. 37.) to be $\frac{nax^{m+n+1}}{m+n+1}$; hence, the equation of the fluents is $x^ny^n = \frac{nax^{m+n+1}}{m+n+1}$.

Ex. 2. Let $\ddot{x} - x\dot{z}^2 = f\dot{z}^2$. As $\ddot{z}$ does not enter into this equation, conceiving it to be deduced from a fluent, $\dot{z}$ must have been supposed constant. Multiply by $\dot{x}$, and $\dot{x}\ddot{x} - x\dot{x}\dot{z}^2 = f\dot{x}\dot{z}^2$, and as $\dot{z}$ is constant, the fluent is $\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2\dot{z}^2 = fx\dot{z}^2$; hence, $\dot{z} = \frac{\dot{x}}{\sqrt{2fx+x^2}}$, whose fluent (Art. 45.) is $z =$ h. l. $(f+x+\sqrt{2fx+x^2})$.

RULE 2.

Sometimes the fluent may be found, by the addition of a new variable quantity.

EXAMPLE.

Let $a\dot{z} = z\dot{x} - x\dot{x}$. Assume $z = a + x + v$, then $\dot{z} = \dot{x} + \dot{v}$; hence, by substitution, $a\dot{x}+a\dot{v} = a\dot{x}+x\dot{x} + v\dot{x} - x\dot{x}$, therefore $a\dot{v} = v\dot{x}$, or $\dot{x} = \frac{a\dot{v}}{v}$; hence, (Art. 45.) $x = a \times$ h. l. v; consequently $z = a + v + a \times$ h. l. v, and by substituting for v it's value $z - a - x$, we get $x = a \times$ h. l. $(z - a - x)$.

RULE 3.

The fluent may sometimes be found, by first putting *the equation into fluxions, making one of the fluxions constant.*

EXAMPLE.

Let $\frac{a\dot{x}+y\dot{x}}{\dot{y}} = x+y-\frac{x\dot{y}}{\dot{x}}$. Make $\dot{y}$ constant, and put the equation into fluxions, and $\frac{(a+y)\times\ddot{x}}{\dot{y}}+\dot{x} = \dot{x}+\dot{y}+\frac{x\dot{y}\ddot{x}-\dot{x}^2\dot{y}}{\dot{x}^2}$; hence, $\frac{(a+y)\times\ddot{x}}{\dot{y}} = \frac{x\dot{y}\ddot{x}}{\dot{x}^2}$, and $(a+y)\times\dot{x}^2 = x\dot{y}^2$, consequently $x^{-\frac{1}{2}}\dot{x} = \overline{a+y}|^{-\frac{1}{2}}\dot{y}$; hence, (Art. 37. and 39.) we have $2x^{\frac{1}{2}} = 2\times\overline{a+y}|^{\frac{1}{2}}$.

RULE 4.

If only one of the variable quantities (x *or* y) *enter, substitute for the fluxion of one of them, the fluxion of the other multiplied into a new variable quantity.*

EXAMPLE.

Let $y\dot{y}^3\dot{x} = a\dot{x}^4+2a\dot{x}^2\dot{y}^2+a\dot{y}^4$, where x is wanting. Assume $z\dot{y} = \dot{x}$, and we get $yz\dot{y}^4 = az^4\dot{y}^4+2az^2\dot{y}^4+a\dot{y}^4$, or $yz = az^4+2az^2+a$; hence, $y = az^3+2az+\frac{a}{z}$, therefore $\dot{y} = 3az^2\dot{z}+2a\dot{z}-\frac{a\dot{z}}{z^2}$, consequently $\dot{x} = z\dot{y} = 3az^3\dot{z}+2az\dot{z}-\frac{a\dot{z}}{z}$, whose fluent is $x = \frac{3}{4}az^4+az^2-a\times$ h. l. z; and if in this equation we substitute the value of z in terms of y, found from the equation $y = az^3+2az+\frac{a}{z}$, we shall get x in terms of y.

PROP. CI.

In any fluxional equation of the second order, where the fluxion of one of the variable quantities (x) *is* *constant, to transform it into one in which* $\dot{y}$ *is constant.*

(159.) Suppose the value of y to be expressed by $a+bx+cx^2+dx^3+$ &c. then $\frac{\dot{y}}{\dot{x}}=b+2cx+3dx^2+$ &c. Make $\dot{x}$ constant, and take the fluxion, and $\frac{\ddot{y}}{\dot{x}}=2c\dot{x}+6dx\dot{x}+$&c. Now make $\dot{y}$ constant, and $\frac{-\dot{y}\ddot{x}}{\dot{x}^2}=2c\dot{x}+6dx\dot{x}+$ &c. when therefore $\dot{x}$ is constant, the value of $\frac{\ddot{y}}{\dot{x}}$ is the same as $\frac{-\dot{y}\ddot{x}}{\dot{x}^2}$ when $\dot{y}$ is constant.

Hence we have the following

RULE 5.

If in any fluxional equation of the second order, in which $\dot{x}$ *is constant, we substitute for* $\frac{\ddot{y}}{\dot{x}}$ *the quantity* $\frac{-\dot{y}\ddot{x}}{\dot{x}^2}$, *or for* $\ddot{y}$ *the quantity* $\frac{-\dot{y}\ddot{x}}{\dot{x}}$, *we shall transform the equation into one in which* $\dot{y}$ *is constant, and thus the fluent may be often found.*

EXAMPLE.

Let $\dot{x}\dot{y}-x\ddot{y}-a\ddot{y}-\frac{x\dot{y}^2}{b}=0$, which being supposed to have arisen from some fluent, $\dot{x}$ is constant, as $\ddot{x}$ does not enter. Substitute $\frac{-\dot{y}\ddot{x}}{\dot{x}}$ for $\ddot{y}$ (in which case $\dot{y}$ becomes constant), and we get $\dot{x}\dot{y}+x\times\frac{\dot{y}\ddot{x}}{\dot{x}}+a\times\frac{\dot{y}\ddot{x}}{\dot{x}}-$

$\frac{x\dot{y}^2}{b} = 0$, or $\dot{x}^2 + x\ddot{x} + a\ddot{x} - \frac{x\dot{x}\dot{y}}{b} = 0$, whose fluent is $x\dot{x} + a\dot{x} - \frac{x^2\dot{y}}{2b}$, which, as the fluxion is = 0, must be equal to some constant quantity; let it be $c\dot{y}$*, and then $\dot{y} = \frac{2bx\dot{x}}{2bc + x^2} + \frac{2ab\dot{x}}{2bc + x^2}$, whose fluents (Art. 45. and 46.) are $y = b \times L + a \times \sqrt{\frac{2b}{c}} \times A$, where A is a circular arc, whose radius is 1 and tangent $\frac{x}{\sqrt{2bc}}$, and $L =$ h. l. $(2bc + x^2)$.

RULE 6.

If the fluxion of one of the unknown quantities $(\dot{x})$ *be constant, and the equation contain* $\dot{x}$, $\dot{y}$, $\ddot{y}$, *and* x *or* y *be wanting, it may be reduced to first fluxions by substituting* $z\dot{x} = \dot{y}$.

Let $\dot{x}^2 + \dot{y}^2 = \frac{y\ddot{y}}{n}$; put $\dot{y} = z\dot{x}$, then $\ddot{y} = \dot{z}\dot{x}$; hence, $n\dot{x}^2 \times (1 + z^2) = y\dot{z}\dot{x}$, or $y\dot{z} = n\dot{x} \times (1 + z^2) = n\dot{y} \times \frac{1 + z^2}{z}$, and $\frac{2n\dot{y}}{y} = \frac{2z\dot{z}}{1 + z^2}$; hence, (Art. 109.) $y^{2n} = (1 + z^2) \times a^2$, or $\frac{y^{2n}}{a^2} - 1 = z^2 = \frac{\dot{y}^2}{\dot{z}^2}$, and $\dot{x} = \frac{a\dot{y}}{\sqrt{y^{2n} - a^2}}$, a denoting an invariable quantity.

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* The given fluxion being supposed to have arisen from some fluent, it is easy to conceive that this constant quantity must be such as $c\dot{y}$; because the equation, after taking the fluent the first time, arose from taking the fluxion of the fluential equation, and therefore $\dot{x}$ or $\dot{y}$ must necessarily enter into every term.

RULE 7.

Sometimes it is useful to substitute for the ratio of the unknown quantities.

EXAMPLES.

1. Let $x\dot{x} + ay\dot{z} + y\dot{y} = 0$. Put $z = \frac{y}{x}$, then $\dot{y} = x\dot{z} + z\dot{x}$; hence, the fluxion becomes $x\dot{x} + azx\dot{x} + z^2x\dot{x} + x^2z\dot{z} = 0$, and $\frac{\dot{x}}{x} = \frac{-z\dot{z}}{z^2+az+1}$, whose fluents are before given.

2. Let $ax\dot{y} + \dot{x}\sqrt{x^2+y^2} = 0$. Put $\frac{x}{y} = \sqrt{z^2 - 1}$; then we get $\frac{\dot{y}}{y} + \frac{z^2\dot{z}}{(z+a) \times (z+1) \times (z-1)} = 0$. Resolve the latter term into $\frac{P}{z+a} + \frac{Q}{z+1} + \frac{R}{z-1}$, and then each fluent is found by the preceding Rules.

RULE 8.

Sometimes a fluent may be found, by assuming a fluent which, when put into fluxions, shall be of the form of the given fluxion, and then finding the values of the assumed constant quantities.

EXAMPLES.

1. Let $(ax+by) \times \dot{x} + (cx+dy) \times \dot{y} = 0$; assume the fluent to be $\overline{x+my}|^n \times \overline{x+ry}|^s = A$ a constant quantity; then $n \times$ h. l. $(x+my) + s \times$ h. l. $(x+ry) =$ h. l. A, and $\frac{n\dot{x}+nm\dot{y}}{x+my} + \frac{s\dot{x}+sr\dot{y}}{x+ry} = 0$; hence, $(n+s) \times x\dot{x} + (sr+sm) \times y\dot{x} + (nm+sr) \times x\dot{y} + (mrn+mrs) \times y\dot{y} = 0$, which is of the form of the given fluxion; and

equating the coefficients of the corresponding terms, we get m, n, r, s, and thence the fluent required.

2. Let $2x\dot{y} + 3y\dot{x} + 10y\dot{y} = 0$, where the term $x\dot{x}$ is wanting. In this case, assume $\overline{x+ay}|^{m} \times y^{n} = A$ a constant quantity; then $m \times$ h. l. $(x + ay) + n \times$ h. l. $y =$ h. l. A, and $\frac{m\dot{x} + ma\dot{y}}{x+ay} + \frac{n\dot{y}}{y} = 0$, or $my\dot{x} + nx\dot{y} + (m+n) \times ay\dot{y} = 0$; compare this with the given equation, and $m=3$, $n = 2$, $a = 2$; hence, the fluent is $\overline{x + 2y}|^{3} \times y^{2}$.

When $y\dot{y}$ is wanting, assume $\overline{ax+y}|^{m} \times x^{n} = A$.

If the fluent cannot be obtained by these means, or any other artifices, it may be necessary to have recourse to infinite series (see Art. 111.) in order to express the fluent, in which case it will be very useful to attend to the following

RULE.

Let the quantity whose value is required be assumed equal to some unknown power, n, *of the other quantity, and let that power with it's fluxion or fluxions be substituted for their supposed equals in the given equation.*

Let the least *exponents for an* ascending, *or* greatest *for a* descending *series, of the quantity thus substituted, be made equal to each other, and thence* n *will be found. Or if there happen to be only one or more terms having the least or greatest index, make the coefficient of that term or terms* = 0, *and you get* n.

Substitute this value of n *for* n, *and take the difference between one of the equal exponents, and every other exponent of the same variable quantity.*

To these differences, write down all the least numbers which can be composed out of them by continual addition, either to themselves, or to one another, till *you get as many terms as the required series is to be continued to.*

Let each of these terms be increased *by* n *for an* ascending *series, and* decreased *by* n *for a* descending *series, and you have the required exponents.*

In equations where the higher order of fluxions are concerned, the series must be assumed in terms of that quantity which flows uniformly, and that is known by observing which quantity has no second, &c. fluxions.

Ex. 1. Let the equation be $a^2\dot{x}^2+x^2\dot{z}^2-a^2\dot{z}^2=0$, when z is a circular arc whose radius is a and sine x.

Assume z^n for x, then $nz^{n-1}\dot{z}=\dot{x}$, and by substitution, the equation becomes $a^2n^2z^{2n-2}\dot{z}^2+z^{2n}\dot{z}^2-a^2\dot{z}^2=0$, and the indices of z are $2n-2$, $2n$, and 0, for we conceive the last term $a^2\dot{z}^2$ to be $a^2z^0\dot{z}^2$; and putting the two least indices $2n-2$ and 0 equal, we get $n=1$; which substituted for n, the indices become 0, 2, 0, and the differences are 0, 2, and by adding 2 continually, we get the series 0, 2, 4, 6, &c., to which add n, or 1, and we get 1, 3, 5, 7, &c. for the indices. Assume therefore $x=pz+qz^3+rz^5+sz^7+\&c.$ and putting $\dot{z}=1$ to shorten the operation, $\dot{x}=p+3qz^2+5rz^4+7sz^6+\&c.$ and this squared and substituted into the given equation, we get

$$\left.\begin{array}{rrrr} a^2p^2+6a^2pqz^2+ & 10a^2prz^4+ & 14a^2psz^6+\&c. \\ & 9a^2q^2z^4+ & 30a^2prz^6+\&c. \\ +\quad p^2z^2+ & 2pqz^4+ & 2prz^6+\&c. \\ -a^2 & + & q^2z^6+\&c. \end{array}\right\}=0;$$

hence (Art. 110.), $a^2p^2-a^2=0$, $6a^2pq+p^2=0$, $10a^2pr+9a^2q^2+2pq=0$, $14a^2ps+30a^2pr+2pr+q^2=0$; &c. and from the first, $p=1$; therefore $6a^2q+1=0$, and $q=-\frac{1}{6a^2}=-\frac{1}{2.3.a^2}$; hence, $10a^2r=-$

$9a^2q^2-2q=-q\times(9a^2q+2)=-q\times(-\frac{3}{2}+2)=-\frac{q}{2}=$

$\frac{1}{2.3.2a^2}$, therefore $r=\frac{1}{2.3.4.5.a^4}$; also, $14a^2ps=-30$ $a^2qr-2pr-q^2=-3a^2\times-\frac{1}{6a^2}\times\frac{1}{120a^4}-2\times\frac{1}{120a^4}$ $-\frac{1}{36a^4}=\frac{1}{24a^4}-\frac{1}{60a^4}-\frac{1}{36a^4}=-\frac{1}{360a^4}$, therefore s $=-\frac{1}{14\times360a^6}=-\frac{1}{2.3.4.5.6.7a^6}$; hence, $x=z-$ $\frac{z^3}{2.3a^2}+\frac{z^5}{2.3.4.5a^4}-\frac{z^7}{2.3.4.5.6.7a^6}+$ &c.

Ex. 2. Let $2ax\dot{y}^2-ay^2\ddot{x}+2x^2\dot{y}^2-2y^2\dot{x}^2=0$. Assume $x=y^n$, and $\dot{x}=ny^{n-1}$, and ($\dot{y}$ being constant) $\ddot{x}=n.(n-1).y^{n-2}\dot{y}^2$; therefore the equation becomes (omitting $\dot{y}^2$) $2ay^n-n.(n-1).ay^n+2y^{2n}-2n^2y^{2n}=0$; here there is only one power of y having the least index, therefore we must assume $2a-n.(n-1).a=0$, or $n.(n-1)=2$, and $n=2$, and this is for an *ascending* series. Substitute this for n, and the indices become, 2, 2, 4, 4; now the difference between one of the least indices 2, and the other indices is 0, 2, and by adding 2 continually, we get the series 0, 2, 4, 6, &c. and increasing these by n, or 2, we get 2, 4, 6, 8, &c. for the required coefficients. Assume therefore $x=$ $py^2+qy^4+ry^6+sy^8+$&c. then $\dot{x}=2py+4qy^3+6ry^5$ $+8sy^7+$&c. (assuming $\dot{y}=1$), and $\ddot{x}=2p+12qy^2+$ $30ry^4+56sy^6+$&c. also, $\dot{x}^2=4p^2y^2+16q^2y^6+16pqy^4$ $+$&c. hence, by substitution, we get

$$\left.\begin{array}{rrrrl} 2apy^2 & +2aqy^4 & +2ary^6 & +2asy^8 & +\text{\&c.}\\ -2apy^2 & -12aqy^4 & -30ary^6 & -56asy^8 & +\text{\&c.}\\ & +2p^2y^4 & +4pqy^6 & +2q^2y^8 & \\ & & & +4pry^8 & +\text{\&c.}\\ & -8p^2y^4 & -32pqy^6 & -32q^2y^8 & \\ & & & -48pry^8 & +\text{\&c.} \end{array}\right\}=0;$$

hence, $2ap-2ap=0$; $2aq-12aq+2p^2-8p^2=0$; $2ar-30ar+4pq-32pq=0$; $2as-56as+2q^2+$ $4pr-32q^2-48pr=0$; from the first equation it appears that p may be assumed at pleasure; from the second equation, $q = \frac{-3p^2}{5a}$; from the third, $r = \frac{3p^3}{5a^2}$; from the fourth, $s = \frac{-31p^4}{45a^3}$; &c. hence, $x = py^2 - \frac{3p^2}{5a}y^4 + \frac{3p^3}{5a^2}y^6 - \frac{31p^4}{45a^3}y^8$ &c.

For a *descending* series, we make the coefficients of the highest powers of $y=0$, or $2-2n^2=0$, and $n=1$; and the indices become 1, 1, 2, 2, and taking one of the greatest, 2, from all the rest, the remainders are -1 and 0, and by adding -1 continually, we get 0, $-1, -2, -3, -4$, &c. and these increased by n, or 1, give $1, 0, -1, -2, -3$, &c.; hence, assume $x = py + q + ry^{-1} + sy^{-2} +$ &c. and we get, as before,

$$\left.\begin{array}{r} 2apy + 2aq + 2ary^{-1} + \&c. \\ - 2ary^{-1} - \&c. \\ 2p^2y^2 + 4pqy + 2q^2 + 4qry^{-1} + \&c. \\ + 4pr + 4psy^{-1} + \&c. \\ - 2p^2y^2 \qquad\qquad + 4pr + 8psy^{-1} + \&c. \end{array}\right\} = 0;$$

hence, $2p^2 - 2p^2 = 0$; $2ap + 4pq = 0$; $2aq + 2q^2 + 8pr = 0$; $4qr + 12ps = 0$; we may therefore assume p at pleasure, and then $q = -\frac{a}{2}$; $r = \frac{a^2}{16p}$; $s = \frac{a^3}{96p^2}$; &c. therefore $x = py - \frac{a}{2} + \frac{a^2}{16py} + \frac{a^3}{96p^2y^2} +$ &c.

Although this rule may become sometimes impracticable, yet when it can be applied it never takes in any unnecessary terms.

SECT. XII.

ON THE SUMMATION OF SERIES.

PROP. CII.

To find the sum of $1^n x + 2^n x^2 + 3^n x^3 + \&c. \ldots s^n x^s$.

(160.) ASSUME $x + x^2 + x^3 + \&c. \ldots x^s = \frac{x^{s+1} - x}{x - 1}$ $= a$; take the fluxion of both sides, divide by $\dot{x}$, and multiply by x; repeat this operation, and you will raise the powers of the natural numbers an unit every time; hence,

$$1\,x + 2\,x^2 + 3\,x^3 + \&c. \ldots s\,x^s = \frac{x\dot{a}}{\dot{x}} = b;$$

$$1^2 x + 2^2 x^2 + 3^2 x^3 + \&c. \ldots s^2 x^s = \frac{x\dot{b}}{\dot{x}} = c;$$

$$1^3 x + 2^3 x^2 + 3^3 x^3 + \&c. \ldots s^3 x^s = \frac{x\dot{c}}{\dot{x}} = d;$$

Thus we may continue the operation to any power.

PROP. CIII.

To find the sum of $1.2.3\,x + 2.3.4\,x^2 + 3.4.5\,x^3 + \&c.$ $(s-2).(s-1).s\,x^{s-2}$.

(161.) Assume as before, take the fluxion, and divide by $\dot{x}$, repeat this operation till you have gotten the number of factors, and then multiply by x; hence,

$$1 + 2x + 3x^2 + 4x^3 + \&c. \ldots\ldots s\,x^{s-1} = \frac{\dot{a}}{\dot{x}} = b;$$

$$1.2+2.3x+3.4x^2+\&c.\ldots(s-1).sx^{s-2}=\frac{\dot{b}}{\dot{x}}=c;$$

$$1.2.3x+2.3.4x^2+\&c.(s-2).(s-1).sx^{s-2}=\frac{x\dot{c}}{\dot{x}}=d.$$

PROP. CIV.

Given $ax^n+bx^{2n}+cx^{3n}+\&c.+mx^{vn}=A$; *to find* $(p+n)\times(q+n)\times ax^n+(p+2n)\times(q+2n)\times bx^{2n}+\&c.\ldots\ldots(p+vn)\times(q+vn).mx^{vn}$.

(162.) Multiply the given equation by x^p, and $ax^{p+n}+bx^{p+2n}+\&c.=Ax^p=B$; take the fluxion and divide by $\dot{x}$, and $(p+n)\times ax^{p+n-1}+(p+2n)\times bx^{p+2n-1}+\&c.=\frac{\dot{B}}{\dot{x}}$; divide by x^{p-1}, and $(p+n)\times ax^n+(p+2n)\times bx^{2n}+\&c.=\frac{\dot{B}}{x^{p-1}\dot{x}}=C$. Now multiply this equation by x^q, take the fluxion, and divide by $x^{q-1}\dot{x}$, and we get $(p+n)\times(q+n)\times ax^n+(p+2n)\times(q+2n)\times bx^{2n}+\&c.=\frac{\dot{\overline{Cx^q}}}{x^{q-1}\dot{x}}$.

In this manner, any factors may be introduced, by multiplying by such powers of x as shall produce the factors required.

PROP. CV.

Let the sum of $\frac{2x}{1}-\frac{4x^3}{3}+\frac{6x^5}{5}-\&c.$ ad infinitum *be required.*

(163.) By Art. 54. Ex. 5. $\frac{x}{1}-\frac{x^3}{3}+\frac{x^5}{5}-\&c.=A$, A being an arc of a circle whose radius $=1$, tangent $=x$. Multiply by x, and $\frac{x^2}{1}-\frac{x^4}{3}+\frac{x^6}{5}-\&c.=Ax$; hence,

$\frac{2x}{1} - \frac{4x^3}{3} + \frac{6x^5}{5} - \&c. = \frac{A\dot{x} + x\dot{A}}{\dot{x}}$ = (because $\dot{A} = \frac{\dot{x}}{1+x^2}$ by Art. 46.) $\dot{A} + \frac{x}{1+x^2}$.

If $x = 1$, then $\frac{2}{1} - \frac{4}{3} + \frac{6}{5} - \&c. = A + \frac{1}{2}$.

Prop. CVI.

To sum series by means of the fluent of $\mathrm{v}\mathrm{x}^{\mathrm{n}}\dot{\mathrm{x}}$, v *being* $= h.\,l.\,\frac{1}{1-\mathrm{x}}$.

(164.) By Art. 153. the fluent of $vx^n\dot{x}$ is $\frac{vx^{n+1}}{n+1} + \frac{1}{n+1} \times \left(\frac{x^{n+1}}{n+1} + \frac{x^n}{n} + \frac{x^{n-1}}{n-1}\right) + \&c. - v = v \times \left(\frac{x^{n+1}}{n+1} - \frac{1}{n+1}\right) + \frac{x^{n+1}}{(n+1)\times(n+1)} + \frac{x^n}{(n+1)\times n} + \frac{x^{n-1}}{(n+1).(n-1)} + \&c.$ But $v = \text{hyp. log.}\,\frac{1}{1-x} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \&c.$ *ad infinit.* hence, $vx^n\dot{x} = x^{n+1}\dot{x} + \frac{1}{2}x^{n+2}\dot{x} + \frac{1}{3}x^{n+3}\dot{x} + \frac{1}{4}x^{n+4}\dot{x} + \&c.$ whose fluent is $\frac{x^{n+2}}{n+2} + \frac{x^{n+3}}{2.(n+3)} + \frac{x^{n+4}}{3.(n+4)} + \frac{x^{n+5}}{4.(n+5)} + \&c.$ Make these two fluents equal, and we have $\frac{v}{n+1} \times (x^{n+1} - 1) + \frac{x^{n+1}}{(n+1)\times(n+1)} + \frac{x^n}{(n+1)\times n} + \frac{x^{n-1}}{(n+1)\times(n-1)} + \&c.$ to $n+1$ terms $= \frac{x^{n+2}}{n+2} + \frac{x^{n+3}}{2.(n+3)} + \frac{x^{n+4}}{3.(n+4)} + \&c.$ *ad infinitum.*

(165.) If $n = 0$, then $\frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{3.4} + \&c.$ *ad infinit.* $= v \times (x-1) + x$; hence, if $x = 1$, $\frac{1}{2} + \frac{1}{2.3} + \frac{1}{3.4} + \&c. = 1$.

(166.) Since $\frac{x^2}{1.2}+\frac{x^3}{2.3}+\frac{x^4}{3.4}+\text{\&c.}=vx-v+x$, multiply by $\dot{x}$, and $\frac{x^2\dot{x}}{1.2}+\frac{x^3\dot{x}}{2.3}+\frac{x^4\dot{x}}{3.4}+\text{\&c.}=vx\dot{x}-v\dot{x}+x\dot{x}$; now by Art. 153. the fluent of $vx\dot{x}$ is $\frac{1}{2}vx^2-\frac{1}{2}v+\frac{1}{4}x^2+\frac{1}{2}x$; also, the fluent of $v\dot{x}$ is $vx-v+x$; hence, the fluent of $vx\dot{x}-v\dot{x}+x\dot{x}$ is $\frac{1}{2}vx^2-vx+\frac{1}{2}v-\frac{1}{2}x+\frac{3}{4}x^2$; consequently (*B*) $\frac{x^3}{1.2.3}+\frac{x^4}{2.3.4}+\frac{x^5}{3.4.5}+\text{\&c.}=\frac{1}{2}vx^2-vx+\frac{1}{2}v-\frac{1}{2}x+\frac{3}{4}x^2$. Assume $\frac{1}{2}vx^2-vx+\frac{1}{2}v=0$, or $x^2-2x+1=0$; hence, $x=1$; make $x=1$, and $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\text{\&c.}=\frac{1}{4}$.

Let $x=\frac{1}{4}$, then $v=\text{h. l. }\frac{4}{3}$; hence, $\frac{1}{1.2.3}\times\frac{1}{4^3}+\frac{1}{2.3.4}\times\frac{1}{4^4}+\text{\&c.}=\frac{9}{32}\times\text{h. l. }\frac{4}{3}-\frac{5}{64}$.

Let $x=\frac{1}{2}$, then $v=\text{h. l. }2$; hence, $\frac{1}{1.2.3}\times\frac{1}{8}+\frac{1}{2.3.4}\times\frac{1}{16}+\text{\&c.}=\frac{1}{8}\times\text{h. l. }2-\frac{1}{16}$. Thus by assuming x and determining v from it, we may find the sum of the corresponding series.

In like manner, by multiplying *B* by $\dot{x}$ and taking the fluent, we shall get four factors in the denominator, 1.2.3.4, 2.3.4.5, &c. or if we multiply by $x\dot{x}$ and take the fluent, we shall get the factors 1.2.3.5, 2.3.4.6, &c. And, in like manner, we may add what factors we please, by multiplying by such a power of x as will produce that factor. If the Reader wish to see more instances, he may consult A. de Moivre's *Miscel. Anal.* Lib. VI.

Prop. CVII.

To sum series from the fluent of $vx^n\dot{x}$, *where* v *is a circular arc, whose radius is unity, and tangent* x.

(167.) By Art. 154. the fluent of $vx^n\dot{x}$ is $\frac{vx^{n+1}}{n+1} + \frac{1}{n+1}\times\left(-\frac{x^n}{n}+\frac{x^{n-2}}{n-2}-\&c.\mp v\right)$, where the sign of v is $+$ or $-$, according as $\frac{n+1}{2}$ is odd or even, when n is an *odd* number. But (Art. 46.) $v = x - \frac{x^3}{3}+\frac{x^5}{5}-\&c.$ hence, $vx^n\dot{x} = x^{n+1}\dot{x} - \frac{x^{n+3}\dot{x}}{3}+\frac{x^{n+5}\dot{x}}{5} - \&c.$ whose fluent is $\frac{x^{n+2}}{n+2}-\frac{x^{n+4}}{3.(n+4)}+\frac{x^{n+6}}{5.(n+6)}-\&c.$ Make these fluents equal, and we have $\frac{1}{n+1}\times\left(vx^{n+1}\mp v - \frac{x^n}{n}+\frac{x^{n-2}}{n-2}-\&c.\right)$ $= \frac{x^{n+2}}{n+2}-\frac{x^{n+4}}{3.(n+4)}+\frac{x^{n+6}}{5.(n+6)}-\&c.$ *ad infinitum.*

Let $\frac{n+1}{2}$ be an *even* number, and assume $vx^{n+1} - v = 0$, and then $x=1$; hence, $\frac{1}{n+1}\times\left(-\frac{1}{n}+\frac{1}{n-2}-\&c.\right)$ to $\frac{n+1}{2}$ terms, is equal to $\frac{1}{n+2}-\frac{1}{3.(n+4)}+\frac{1}{5.(n+6)}-$ &c. *ad infinitum.*

If $n=3$, then $\frac{1}{1.5}-\frac{1}{3.7}+\frac{1}{5.9}-\&c.$ *ad infinitum* $= \frac{1}{4}\times\left(-\frac{1}{3}+1\right)=\frac{1}{6}$.

Let $\frac{n+1}{2}$ be an *odd* number, and assume $n = 1$,

$x=1$; then v becomes an arc of 45°; and we get $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7}$ − &c. *ad infinitum* = arc 45° − ½.

If n be an *even* number, then (Art. 154.) we get, in like manner,

$$\frac{1}{n+1} \times \left(vx^{n+1} - \frac{x^n}{n} + \frac{x^{n-2}}{n-2} - \text{\&c.} \mp \text{h. l.} \sqrt{1+x^2}\right) = \frac{x^{n+2}}{n+2} - \frac{x^{n+4}}{3.(n+4)} + \frac{x^{n+6}}{5.(n+6)} - \text{\&c. } \textit{ad infinitum},$$

where the number of terms to be taken in the first series is $\frac{1}{2}n$, the first and last terms excepted, and the sign of the last term is + or −, according as $\frac{1}{2}n$ is odd or even.

If $n=2$, and $x=1$, then v becomes an arc of 45°; and we get $\frac{1}{1.4} - \frac{1}{3.6} + \frac{1}{5.8}$ − &c. *ad infinitum* = $\frac{1}{3}$ × (arc 45° − ½ + h. l. $\sqrt{2}$). For more upon this subject, see A. de MOIVRE's *Miscel. Anal.* Lib. VI.

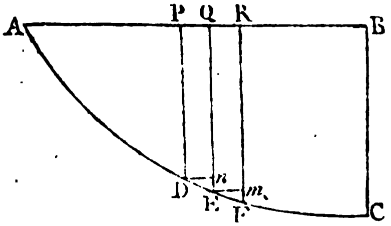
SECT. XIII.

ON THE MAXIMA AND MINIMA OF CURVES.

PROP. CVIII.

To find the nature of curves, in which some quantities remaining invariable, others are the greatest or least possible.

(168.) LET *ABC* be any curvilinear area, *PD, RF* two fixed ordinates indefinitely near to each other, and the ordinate *QE* an arithmetic mean between



them, so that $En = Fm$, Dn, Em being parallel to *AB*. Now it is manifest, that the nature of the curve *DEF* must depend upon the position of the point *E*, as by varying the position of that point, you must necessarily vary the curve; upon the situation therefore of this intermediate ordinate, the determination of the equation to the curve, from the data, will depend. Hence, *PQ, QR*, are the only variable quantities.

(169.) Let any given quantity *M* be made up of *A, B, C, D, E,* &c. or let $A+B+C+D+E+\&c. = M$, and at the same time let some other quantity *m* be required to be a maximum or minimum, and let the corresponding parts of m be a, b, c, d, e, &c. and then will $a+b+c+d+e+$&c. $= m$, M and m being expressed in terms of the same variable quantities. Now let us suppose all the quantities in each to remain constant, except two which correspond. that is, let C and D, c and d be alone variable; then $C+D$ is constant, and to satisfy the other condition, $c+d$ must be a maximum or minimum; hence, (Art. 21.), $\dot{C}+\dot{D}=0$, $\dot{c}+\dot{d}=0$, and from these two equations we may get the relation of the variable quantities which compose them, which will be found sufficient to determine the nature of the curve.

Prop. CIX.

Given the points A *and* C, *to find the curve in which a body will descend from* A *to* C *in the least time possible.*

(170.) Put $PD=m$, $QE=n$, $En=Fm=a$, the constant quantities, $v=PQ=Dn$, $w=QR=Em$; then $DE=\sqrt{a^2+v^2}$, and $EF=\sqrt{a^2+w^2}$. Now AB being parallel to the horizon, the velocities at D, and E are as $\sqrt{m}$ and $\sqrt{n}$, by *Mechanics*; also, the times being as the spaces directly and velocities inversely, the times through DE, EF will be as $\frac{\sqrt{a^2+v^2}}{\sqrt{m}}$ and $\frac{\sqrt{a^2+w^2}}{\sqrt{n}}$; hence, as AB is given, v, w are two parts of this given quantity, whose sum $v+w$ is constant; also, $\frac{\sqrt{a^2+v^2}}{\sqrt{m}}$ and $\frac{\sqrt{a^2+w^2}}{\sqrt{n}}$ are the two corresponding parts of the minimum, whose sum $\frac{\sqrt{a^2+v^2}}{\sqrt{m}}+\frac{\sqrt{a^2+w^2}}{\sqrt{n}}=$ minimum (Art. 169.); hence, $\dot{v}+\dot{w}=0$, and $\frac{v\dot{v}}{\sqrt{m}\times\sqrt{a^2+v^2}} + \frac{w\dot{w}}{\sqrt{n}\times\sqrt{a^2+w^2}} = 0$; ∴ $\dot{w} = -\dot{v}$; consequently $\frac{v\dot{v}}{\sqrt{m}\times\sqrt{a^2+v^2}} - \frac{w\dot{w}}{\sqrt{n}\times\sqrt{a^2+w^2}} = 0$, and $\frac{v}{\sqrt{m}\times\sqrt{a^2+v^2}} = \frac{w}{\sqrt{n}\times\sqrt{a^2+w^2}}$; now these are two similar quantities, which express (in their ultimate state) the fluxion of the abscissa divided by the square root of the ordinate × fluxion of the curve; two successive values of this quantity therefore being equal to each other, shows the quantity itself to be constant; hence, put $AP=x$, $PD=y$, $AD=z$, and we have $\frac{\dot{x}}{\sqrt{y}\times\dot{z}} = \frac{1}{\sqrt{r}}$ a constant quantity, which is the property of a cycloid, the diameter of whose generating semicircle is r.

Prop. CX.

To determine the nature of the curve AC *whose length is given, when it's area is a maximum.*

(171.) The same notation remaining, we have $DE+EF=\sqrt{a^2+v^2}+\sqrt{a^2+w^2}$ a constant quantity, the sum of two parts of the given curve line AC; also, $mv+nw$ is the sum of the two corresponding parts of the maximum; hence (Art. 169.), $mv+nw=$ max. ∴ $m\dot{v}+n\dot{w}=0$, and $\frac{v\dot{v}}{\sqrt{a^2+v^2}}+\frac{w\dot{w}}{\sqrt{a^2+w^2}}=0$; hence, $\frac{m\dot{v}}{n}=-\dot{w}$, therefore $\frac{v\dot{v}}{\sqrt{a^2+v^2}}-\frac{mw\dot{v}}{n\sqrt{a^2+w^2}}=0$, consequently $\frac{v}{m\sqrt{a^2+v^2}}=\frac{w}{n\sqrt{a^2+w^2}}$; which being similar quantities, we have $\frac{\dot{x}}{y\dot{z}} = \frac{1}{r}$ a constant quantity, or $r\dot{x} = y\dot{z}$ the equation of a circle by Art. 46.

PROP. CXI.

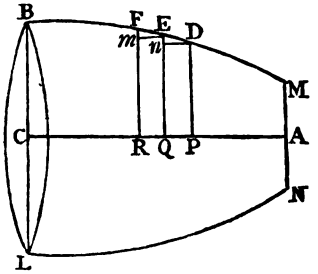
Let the surface of the solid generated by the revolution of the curve AC *about* AB *be given; to find the nature of the curve, when the solid is a maximum.*

(172.) Put $p = 3{,}14159$, &c. then (Art. 56.) $2pm \times \sqrt{a^2 + v^2} + 2pn\sqrt{a^2 + w^2} =$ the sum of the two parts of the given surface generated by $DE + EF$, a constant quantity; also, $pm^2v + pn^2w =$ the sum of the two corresponding parts of the maximum, generated by *PQED*, *QRFE*; hence, $pm^2v + pn^2w =$ max. ∴ (neglecting the constant multiplier p) $m^2\dot{v} + n^2\dot{w} = 0$, and $\frac{mv\dot{v}}{\sqrt{a^2+v^2}} + \frac{nw\dot{w}}{\sqrt{a^2+w^2}} = 0$; hence, $\dot{w} = -\frac{m^2\dot{v}}{n^2}$, which substituted for $\dot{w}$ in the second equation, we get $\frac{v}{m\sqrt{a^2+v^2}} = \frac{w}{n\sqrt{a^2+w^2}}$, which are the same quantities as in the last case; hence, the curve is a circle.

PROP. CXII.

To find the nature of the curve which generates a solid of the least resistance, when moving in a fluid in the direction of it's axis, it's greatest diameter BL *and length* AC *being given.*

(173.) By the Principles of Hydrostatics, the resistance against DE is as $\frac{ma^3}{a^2+v^2}$, and against EF as $\frac{na^3}{a^2+w^2}$; hence, the sum of the two parts of the quantity which is to be a minimum $= \frac{ma^3}{a^2+v^2} + \frac{na^3}{a^2+w^2}$; also, as AC is given, $v+w$, the sum of the two corresponding parts of the given quantity, is constant; therefore $-\frac{2ma^3v\dot{v}}{(a^2+v^2)^2}-\frac{2na^3w\dot{w}}{(a^2+w^2)^2}=0$, and $\dot{v}+\dot{w}=0$; hence, $\dot{v}=-\dot{w}$; consequently, by substitution, $\frac{ma^3v}{(a^2+v^2)^2}=\frac{na^3w}{(a^2+w^2)^2}$, which being similar quantities, we have $\frac{y\dot{y}^3\dot{x}}{\dot{z}^4}=r$ a given quantity, which is the fluxional equation of the curve.



That the curve does not meet the axis at *A*, appears from hence; $y=r\times\frac{\dot{z}^4}{\dot{y}^3\dot{x}}=r\times\frac{ED^4}{En^3\times Dn}$, where the numerator must evidently be greater than the denominator, and therefore y must be greater than r.

(174.) If the greatest diameter *BL*, and *area BMNL* be given, then $mv+nw$ will be given, consequently $m\dot{v}+n\dot{w}=0$, which gives $\frac{\dot{y}^3\dot{x}}{\dot{z}^4}=r$, the equation of the curve.

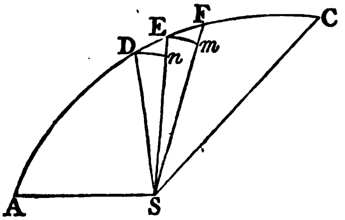
If the greatest diameter and *bulk* be given, then instead of $v+w$ being given, pm^2v+pn^2w will be given (Art. 169.); hence, $m^2\dot{v}+n^2\dot{w}=0$, which gives $\frac{\dot{y}^3\dot{x}}{y\dot{z}^4}=r$, the equation of the curve.

Although *PDEQ*, *QEFR* are here taken as increments, yet we reason upon them as fluxions, conceiving their limiting ratio to be taken, and consequently the conclusions are mathematically true.

Prop. CXIII.

To find the nature of the curve AC, *so that a body may move from* A *to* C *in the least time possible, the velocity at any point* D *being as* DS^r, S *being any fixed point.*

(175.) Let *DS*, *FS*, be two given distances including a given angle *DSF*, draw *SE*, and *Dn* perpendicular



to *SE*, and *Em* to *SF*, and let $En = mF$. Put $SD = m$, $SE = n$, $En = Fm = a$, the constant quantities, $Dn = v$, $Em = w$ the variable quantities; then $DE = \sqrt{a^2+v^2}$ and $EF = \sqrt{a^2+w^2}$; and the time of describing $DE = \frac{\sqrt{a^2+v^2}}{m^r}$, and of $EF = \frac{\sqrt{a^2+w^2}}{n^r}$; hence, $\frac{\sqrt{a^2+v^2}}{m^r} + \frac{\sqrt{a^2+w^2}}{n^r} =$ max. and it's fluxion $\frac{v\dot{v}}{m^r\sqrt{a^2+v^2}} + \frac{w\dot{w}}{n^r\sqrt{a^2+w^2}} = 0$; but the $\angle DSE$ is measured by $\frac{v}{m}$, and $\angle ESF$ by $\frac{w}{n}$; therefore $\frac{v}{m} + \frac{w}{n} = \angle DSF$, and $\frac{\dot{v}}{m} + \frac{\dot{w}}{n} = 0$; hence, $\dot{w} = \frac{-n\dot{v}}{m}$, therefore $\frac{v\dot{v}}{m^r\sqrt{a^2+v^2}} - \frac{nw\dot{v}}{mn^r\sqrt{a^2+w^2}} = 0$, and

$\frac{v}{m^{r-1}\sqrt{a^2+v^2}} = \frac{w}{n^{r-1}\sqrt{a^2+w^2}}$; that is, if $SD = x$, $AD = z$, $Dn = \dot{y}$, then $\frac{\dot{y}}{x^{r-1}\dot{z}} = \frac{1}{c^{r-1}}$ a constant quantity.

If $r = 1$, then $\frac{\dot{y}}{\dot{z}}$ is constant, and the curve is the log. spiral.

If $r = 2$, then $c\dot{y} = x\dot{z}$, and the curve is a circle.

There are several other methods of solving this kind of Problems, but perhaps this is the easiest for the young Student, as showing more clearly how the steps bear upon the question. From this, however, another method presents itself.

We have already $\dot{C} + \dot{D} = 0$, $\dot{c} + \dot{d} = 0$. Now consider C to represent, for instance, PQ, and D, QR, which two quantities are denoted by v and w, then C, c, the corresponding parts will be represented in terms of v; for if the second series represent, for instance, the corresponding increments of the curve, then c will represent DE, which $= \sqrt{a^2 + v^2}$, and thus in all other cases; hence, for $\dot{C}$ put $\dot{v}$, and for $\dot{c}$ put $\alpha\dot{v}$; for $\dot{D}$ put $\dot{w}$, and for $\dot{d}$ put $\beta\dot{w}$; we have therefore $\dot{v} + \dot{w} = 0$, $\alpha\dot{v} + \beta\dot{w} = 0$; or $m\dot{v} + m\dot{w} = 0$, and $n\alpha\dot{v} + n\beta\dot{w} = 0$, where m, n, may be any numbers, positive or negative. Hence, $(m + n\alpha) \times \dot{v} + (m + n\beta) \times \dot{w} = 0$; and making the coefficients of each term $= 0$, we have $m + n\alpha = 0$, $m + n\beta = 0$. Now $(m + n\alpha) \times \dot{v} = m\dot{C} + n\dot{c}$ which therefore $= 0$. Let $x = AP$, $y = PD$, $z = AD$; then from what is already done, y is constant, and also $\dot{y}$ (a), and $\dot{x}$ only is variable, and represents v; hence, $\dot{v} = \ddot{x}$; which in the equation $(m + n\alpha) \times \ddot{x} = 0$, may be omitted, and we may put $m + n\alpha = 0$. Thence, the following

RULE.

Multiply the correspondent fluxions $(\dot{C}, \dot{c})$ *by* m *and* n, *and make the sum* = 0, *and you get the equation of the curve.*

You may put one of the quantities m or n, = 1, and the other negative, in the case of two quantities, as generally more convenient.

Ex. 1. *Given the abscissa, to find the curve, when the time of descent upon it is a minimum.*

Put $x = AP$, $y = PD$, $z = AD$; then, C represents $\dot{x}$, and the time down $\dot{z}$ is as $\frac{\dot{z}}{y^{\frac{1}{2}}} = \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{y^{\frac{1}{2}}}$, which represents c; hence, (if $m = 1$) $\ddot{x} - \frac{n\dot{x}\ddot{x}}{y^{\frac{1}{2}}\sqrt{\dot{x}^2 + \dot{y}^2}} = 0$, and $y^{\frac{1}{2}}\dot{z} = n\dot{x}$, which is the property of the cycloid.

Ex. 2. *Let the length of the curve be given, to find the area a maximum.*

Here $C = \dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$ is given, and $c = y\dot{x}$; hence, $\frac{m\dot{x}\ddot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} - y\ddot{x} = 0$, and $m\dot{x} = y\dot{z}$ the property of a circle by Art. 46.

If the area had been given to find the length a minimum, the same conclusion would have followed.

Ex. 3. *Let the abscissa be given, and the surface a maximum; to find the curve.*

Here $C = \dot{x}$, $c = 2py\sqrt{\dot{x}^2 + \dot{y}^2}$; hence, (leaving out the $2p$), $m\ddot{x} - \frac{y\dot{x}\ddot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = 0$, and $m\dot{z} = y\dot{x}$, the property of the circle.

We may also apply this to spirals. Let us take the Example in page 205. Put $y = SD$, $\dot{x} = Dn$, $\dot{z} = DE$;

then $C=\frac{\dot{x}}{y'}$, $c=\frac{\sqrt{\dot{x}^2+\dot{y}^2}}{y'}$; hence, $\frac{m\ddot{x}}{y}-\frac{\dot{x}\,\ddot{x}}{y'\sqrt{\dot{x}^2+\dot{y}^2}}=0$, and $\frac{\dot{x}}{y'^{-1}\dot{z}}=m$ a constant quantity.

If instead of supposing a maximum or minimum, and *one* given quantity, we suppose *more* given quantities, then we must conceive each quantity to be in like manner divided into correspondent parts. For instance, let there be two given quantities; then this new given quantity we represent by $a'+b'+c'+d'+e'$ +&c. and hence, by proceeding as before, $m\dot{C}+n\dot{c}+p\dot{c}'=0$. Here we interpose two ordinates between *PD*, *RF*, and besides *v*, *w*, we shall have another increment *t*, and so on.

Ex. *Given the abscissa, the length of the curve, and the time down the curve a minimum, to find the curve.*

Here $C=\dot{x}$, $c=\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}$, and $c'=\frac{\dot{z}}{y^{\frac{1}{2}}}=\frac{\sqrt{\dot{x}^2+\dot{y}^2}}{y^{\frac{1}{2}}}$ a maximum. Hence, (leaving out $\ddot{x}$) $m-\frac{n\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}+\frac{p\dot{x}}{y^{\frac{1}{2}}\sqrt{\dot{x}^2+\dot{y}^2}}=0$; or $(ny^{\frac{1}{2}}-p)^2\times\dot{x}^2=m^2y\times(\dot{x}^2+\dot{y}^2)$ the equation of the curve,

In any curve referred to an abscissa and ordinate, $\dot{x}$, $y\dot{x}$, $\dot{z}\,(\sqrt{\dot{x}^2+\dot{y}^2})$, $2py\sqrt{\dot{x}^2+\dot{y}^2}$, $py^2\dot{x}$, represent the fluxion of the abscissa, area, curve line, surface of the solid, and solid respectively; let therefore these considered as given quantities, be represented by *C*, *c*, c', c'', c''', and let *M* represent the maximum or minimum; then by the same reasoning it appears, that $m\dot{C}+n\dot{c}+p\dot{c}'+q\dot{c}''+r\dot{c}'''+\dot{M}=0$. Or without *M*, we may suppose one or more of the other quantities to be given, and some one a maximum or minimum. Thus we get the equation of the curve.

It is supposed that x does not enter into any of the quantities here employed; it does not enter into C, c, c', c'', c'''; if therefore it does not enter into M, the above equation will hold good; when x enters into M it is not so easy to arrive at a general rule, although particular cases may be frequently solved.

There is another method of solving Problems of this kind, which we shall here give, and the Reader may take his choice.

Prop. CXIV.

Let $\dot{y}$ *be given; then when* $A\dot{z} - B\dot{x}$ *is a maximum or minimum,* $A\dot{x} = B\dot{z}$.

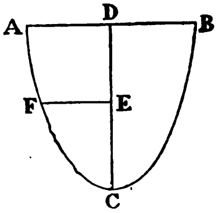
Here $A\dot{z} - B\sqrt{\dot{z}^2 - \dot{y}^2}$ is a max. or min. hence (putting $\dot{x}$ for $\sqrt{\dot{z}^2 - \dot{y}^2}$) $A\ddot{z} - \dfrac{B\dot{z}\ddot{z}}{\dot{x}} = 0$, therefore $A\dot{x} = B\dot{z}$.

Ex. 1. *Given the points* A, C, *and the length of the curve* AC; *to find when the area* ABC *is a maximum.* (Fig. Prop. 108.)

As A and C are given, the point B is given; hence, the fluent of $\dot{x}$ is given, or of $a\dot{x}$; also, the fluent of $\dot{z}$, or of $b\dot{z}$, is given; and the fluent of $y\dot{x}$ or of $cy\dot{x}$ is a max. hence, $b\dot{z} - (a + cy) \times \dot{x} =$ max. it being the addition of constant quantities to that quantity which is a max. Therefore $b\dot{x} = (a + cy) \times \dot{z}$, an equation to a circle. The quantities a, b, c, are determined from the given circumstances.

Ex. 2. *Given the base* AB, *and length of the curve* ACB; *to find the curve, when the centre of gravity* *lies at the lowest point possible from the horizontal base* AB.

Let *DC* be the axis, $CE = x$, $EF = y$, $CF = z$; then the fluent of $a\dot{y}$ is given, the fluent of $b\dot{z}$ is given,



and (Prop. 27.) if $l =$ the length of the curve, the distance of the centre of gravity below $D = \frac{\text{flu. } x\dot{z}}{l} =$ a min. or fluent $cx\dot{z} = \text{min.}$; hence, $a\dot{y} - (b + cx) \times \dot{z} = \text{min.}$ therefore $a\dot{z} = (b + cx) \times \dot{y}$; but when $x = 0$, $\dot{y} = \dot{z}$, and hence, $a = b$, and putting $c = 1$, $a\dot{z} = (a + x) \times \dot{y}$ an equation to the catenary.

The reader will perceive that here, y takes place of x, and x of y.

Ex. 3. *The points* A, C, *being given, and length of the curve, to find when the solid generated about* AB *is a maximum.*

Here the fluent of $a\dot{x}$, $b\dot{z}$ are given, and (Prop. 22.) the fluent of $py^2\dot{x}$, or $cy^2\dot{x}$ is a max.; hence, $b\dot{z} - (a + cy) \times \dot{x} = \text{max.}$ therefore $b\dot{x} = (a + cy^2) \times \dot{z}$ an equation to the *elastic curve*.

It is here understood that *a*, *b*, *c*, denote coefficients either positive, negative, or may be made to vanish; and that the max. or min. has other given quantities annexed to it, and therefore such a quantity still remains a max. or min. To express the equation more generally, we may assume $(a+by+cy^2+\&c.) \times \dot{x} = (p+qy+ry^2+\&c.) \times \dot{z}$. By this method there is no occasion to have recourse to fluxions of an higher order, or to resolve the curve into a number of indefinitely small parts.

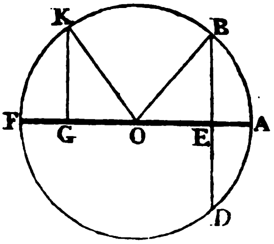
SECT. XIII.

MISCELLANEOUS PROPOSITIONS.

PROP. CXV.

Given the sine EB *of an arc* AB *of a circle; to find the sine of* n *times* AB.

(176.) LET $AB = z$, and $AK = nz$; put $OB = 1$, $y = OE$ the cosine of AB, $v =$ the sine $BE = \sqrt{1^2 - y^2}$, $x =$ the cosine OG of AK, then $\sqrt{1^2 - x^2} =$ the sine GK of AK. Now (Art. 46.) $\dot{z} : -\dot{y} :: 1 : \sqrt{1^2 - y^2}$, $\therefore \dot{z} = \frac{-\dot{y}}{\sqrt{1^2 - y^2}}$; for the same reason, the fluxion of nz, or $n\dot{z} = \frac{-\dot{x}}{\sqrt{1^2 - x^2}}$; hence, $\frac{\dot{x}}{\sqrt{1^2 - x^2}} = \frac{n\dot{y}}{\sqrt{1^2 - y^2}}$; multiply both denominators by $\sqrt{-1}$, and $\frac{\dot{x}}{\sqrt{x^2 - 1^2}} =$



$\frac{n\dot{y}}{\sqrt{y^2-1^2}}$, whose fluent (Art. 45.) is h. l. $(x+\sqrt{x^2-1^2})$ $= n \times$ h. l. $(y+\sqrt{y^2-1^2})$; hence, (Art. 109.) $x+\sqrt{x^2-1^2} = \overline{y+\sqrt{y^2-1^2}}|^n =$ (Art. 34.) $y^n + ny^{n-1}\sqrt{y^2-1^2} + n.\frac{n-1}{2}.y^{n-2}\times(y^2-1^2) + n.\frac{n-1}{2}.\frac{n-2}{3}.y^{n-3}\times\sqrt{y^2-1^2}\times(y^2-1^2)$ + &c. Now as this equation consists of quantities, partly possible and partly impossible, $\sqrt{x^2-1^2}$ and $\sqrt{y^2-1^2}$, being impossible, it is manifest that the possible and impossible parts must be respectively equal. Hence, assuming the *impossible* parts equal, we have, $\sqrt{x^2-1^2} = ny^{n-1}\sqrt{y^2-1^2} + n.\frac{n-1}{2}.\frac{n-2}{3}y^{n-3}\times\sqrt{y^2-1^2}\times(y^2-1^2)$ + &c. Multiply both sides by $\sqrt{-1}$, and $\sqrt{1^2-x^2} = ny^{n-1}\sqrt{1^2-y^2} + n.\frac{n-1}{2}.\frac{n-2}{3}y^{n-3}\times\sqrt{1^2-y^2}\times(y^2-1^2)$ + &c. = (because $v = \sqrt{1^2-y^2}$, and $-v^2 = y^2-1^2$) $ny^{n-1}v - n.\frac{n-1}{2}.\frac{n-2}{3}y^{n-3}v^3$ + &c. the sine of *AK*.

PROP. CXVI.

Given as before, to find the cosine of AK.

(177.) Assume the *possible* parts of the above equation equal, and we have $x = y^n + n.\frac{n-1}{2}y^{n-2}\times(y^2-1^2)$ + &c. $= y^n - n.\frac{n-1}{2}y^{n-2}v^2$ + &c. the cosine of *AK*.

PROP. CXVII.

Given as before, to find the tangent of AK.

(178.) Put t = tangent of AB, then by Plane Trig. $t = \frac{v}{y}$, radius being unity; hence, the tangent of $AK =$

$$\frac{\sin. AK}{\cos. AK} = \frac{n y^{n-1} v - n.\frac{n-1}{2}.\frac{n-2}{3}.y^{n-3} v^3 + \&c.}{y^n - n.\frac{n-1}{2} y^{n-2} v^2 + \&c.} = \text{(by}$$

dividing the numerator and denominator by y)

$$\frac{\frac{nv}{y} - n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{v^3}{y^3} + \&c.}{1 - n.\frac{n-1}{2}.\frac{v^2}{y^2} + \&c.} = \frac{nt - n.\frac{n-1}{2}.\frac{n-2}{3}.t^3 + \&c.}{1 - n.\frac{n-1}{2}.t^2 + \&c.}$$

PROP. CXVIII.

To resolve $v^{2n} - 2xv^n + 1^{2n} = 0$ *into it's quadratic divisors, the limits of* x *being* + 1 *and* − 1.

(179.) Retaining the notation in Art. 176, we have $x + \sqrt{x^2 - 1^2} = \overline{y + \sqrt{y^2 - 1^2}}\,|^n$. Put $v = y + \sqrt{y^2 - 1^2}$; transpose y and square both sides, and we get $v^2 - 2yv = -1^2$, $\therefore v^2 - 2yv + 1^2 = 0$. Also, $v^n = x + \sqrt{x^2 - 1^2}$; hence, by transposing x, and proceeding as before, we get $v^{2n} - 2xv^n + 1 = 0$, the given equation, of which we have one quadratic divisor $v^2 - 2yv + 1^2 = 0$, v being the same in both equations. Now if to the arc AK, we add 360°, 2 × 360°, &c. we shall come again to the same point K, and consequently we shall have the same cosine, or x; hence, x is the cosine of AK, $360° + AK$, $2 \times 360° + AK$, &c. But y is the cosine of an n^{th} part of that arc whose cosine is x; hence, y is the cosine of $\frac{AK}{n}$, $\frac{360° + AK}{n}$, $\frac{2 \times 360° + AK}{n}$,

&c. which cosines call a, b, c, &c. substitute therefore these values for y in the equation $v^2 - 2yv + 1^2 = 0$, and we get $v^2 - 2av + 1^2 = 0$, $v^2 - 2bv + 1^2 = 0$, $v^2 - 2cv + 1^2 = 0$, &c. for the quadratic divisors required; hence, $(v^2 - 2av + 1^2) \times (v^2 - 2bv + 1^2) \times$ &c. $= v^{2n} - 2xv^n + 1^{2n}$, retaining the power of the radius in the last term. Although there are an infinite number of arcs whose cosines are x, and consequently an infinite number of corresponding values of y, yet there are only n *different* values of y; because, after taking n arcs, $\frac{AK}{n}$, $\frac{360 + AK}{n}$, &c. the same cosines will return again.

If $x = \pm 1$, or if AK be taken equal to the whole circumference, or half the circumference, the equation becomes $v^{2n} \mp 2v^n + 1^{2n} = 0$, whose square root is $v^n \mp 1^n = 0$; now as every equation which is a square, must have to every root another equal to it, the equation $v^n \mp 1^n = 0$ must contain the same roots as $v^{2n} \mp 2v^n + 1^{2n} = 0$; the roots therefore of $v^n \mp 1^n = 0$ are found in like manner.

(180.) Hence, we may find the quadratic divisors of $v^{2n} - 2xr^n v^n + r^{2n} = 0$, which is the equation $v^{2n} - 2xv^n + 1^{2n} = 0$, having it's roots multiplied by r (*Alg.* Art. 286.); multiplying the roots therefore of the above quadratics by r, we have $v^2 - 2arv + r^2 = 0$, $v^2 - 2brv + r^2 = 0$, &c. for the quadratics required. If $AK = 90°$, then $x = 0$, and the equation becomes $v^{2n} + r^{2n} = 0$.

LEMMA.

Let $1 - 2xv^n + v^{2n} = 0$, a recurring equation, have it's roots $m, p, q \ldots \frac{1}{m}, \frac{1}{p}, \frac{1}{q}$; and assume (Art. 179.) $1 - 2xv^n + v^{2n} = (1 - 2av + v^2) \times (1 - 2bv + v^2) \times$ &c. $= (1 - mv) \times (1 - pv) \times (1 - qv) \times$ &c. hence, $mp = 1$,

$m+p=2a$, &c. &c. Now $v^n+\frac{1}{v^n}=2x$; for v put m, and $m^n+\frac{1}{m^n}=2x$, or $m^n+p^m=2x$, where x is the cosine of an arc which is to the arc whose cosine is a, as $n:1$; for the same reason $m^{n-1}+p^{n-1}=2e$, if e be the cosine of an arc which is to the arc whose cosine is a, as $n-1:1$. In like manner, $qr=1$, $q+r=2b$; and putting f in the place of e, $q^m+r^m=2x$, $q^{m-1}+r^{m-1}=2f$; and so on.

PROP. CXIX.

To resolve $\frac{1}{1-2xv^n+v^{2n}}$ *into* $\frac{P-Qv}{1-2av+v^2}+\frac{R-Sv}{1-2bv+v^2}+$ &c. x *being the same as in the last Proposition.*

(181.) As $1-2xv^n+v^{2n}=(v-m)\times(v-p)\times(v-q)\times x$, take the fluxion, divide by v, and make $v=m$, and we get $2nm^{2n-1}-2nxm^{n-1}=(m-p)\times(m-q)\times$ &c. Assume $\frac{1}{1-2xv^n+v^{2n}}=\frac{A}{1-mv}+\frac{B}{1-pv}+\frac{C}{1-qv}+$ &c. $=\frac{A+B-(pA+mB)\times v}{1-2av+v^2}+$ &c. reduce the former to a common denominator, and $A\times(1-pv)\times(1-qv)\times$ &c. $+B\times(1-mv)\times(1-qv)\times$ &c. $+$ &c. $=1$; let $1-mv=0$, and $v=\frac{1}{m}$; hence, $A\times\frac{2nm^{2n-1}-2nxm^{n-1}}{m^{2n-1}}=1$, and $A=\frac{m^n}{2nm^n-2nx}$. For the same reason, $B=\frac{p^n}{2np^n-2nx}$; hence, $A+B=$

$\frac{4nm^n p^n - 2xn \times (m^n + p^n)}{4n^2 m^n p^n - 4nx \times (m^n + p^n) + 4n^2 x^2}$ = (substituting 1 for $m + p$, $2x$ for $m^n + p^n$, and reduction) $\frac{1}{n}$. Also,

$$pA + mB = \frac{2n \times (m+p) \times m^n p^n - 2nxpm \times (m^{n-1} + p^{n-1})}{4n^2 \times (p^n - x) \times (m^n - x)}$$

= (by the lemma and reduction) $\frac{a - ex}{n - nx}$; hence, $\frac{A}{1 - pv} + \frac{B}{1 - qv} = \frac{\frac{1}{n} - \frac{a - ex}{n - nx^2} \times v}{1 - 2av + v^2}$. Consequently $\frac{1}{1 - 2xv^n + v^{2n}}$

$$= \frac{\frac{1}{n} - \frac{a - ex}{n - nx^2} \times v}{1 - 2av + v^2} + \frac{\frac{1}{n} - \frac{b - fx}{n - nx^2} \times v}{1 - 2bv + v^2}$$

+ &c. where f is found from b, in the same manner that e is found from a; and so on.

(182.) If x be negative, the given quantity becomes $\frac{1}{1 + 2xv^n + v^{2n}}$.

(183.) In like manner, $\frac{1}{1 \pm v^n}$ will be found equal to $\frac{A}{1 - mv} + \frac{B}{1 - pv}$ + &c. where $A = \frac{1}{n}$, $B = \frac{1}{n}$, &c. and if $(1 - mv) \times (1 - pv) = 1 - 2av + v^2$, then $\frac{A}{1 - mv} + \frac{B}{1 - pv}$

$$= \frac{\frac{2}{n} - \frac{2a}{n} \times v}{1 - 2av + v^2};$$

and so on; n being an *even* number.

If n be an *odd* number, then of the equation $1 + v^n = 0$, one root $= -1$; hence, $1 + v = 0$ is one of the simple equations; and as the other part is made up of quadratics, we have $\frac{1}{1+v^n} = \frac{\frac{2}{n} - \frac{2a}{n} \times v}{1-2av+v^2} + \&c. + \frac{\frac{1}{n}}{1+v}$.

If n be an *odd* number, the equation $1 - v^n = 0$ contains one simple equation, and $\frac{n-1}{2}$ quadratics. Now the equation $1 - v^n = 0$, has one root $= 1$, consequently the simple equation is $1 - v = 0$. Hence,

$$\frac{1}{1-v^n} = \frac{\frac{2}{n} - \frac{2a}{n} \times v}{1-2av+v^2} + \&c. + \frac{\frac{1}{n}}{1+v}.$$

If n be an *even* number, $1 - v^n = 0$ has two roots, -1, $+1$; therefore two of the simple equations will be $1 - v = 0$, $1 + v = 0$; hence, $\frac{1}{1-v^n} = \frac{\frac{2}{n} - \frac{2a}{n} \times v}{1-2av+v^2}$

$+ \&c. + \frac{\frac{1}{n}}{1-v} + \frac{\frac{1}{n}}{1+v}$.

Prop. CXX.

Let $\dot{F} = \frac{\dot{v}}{1-2xv^n+v^{2n}}$ *to find* F, x *being constant, and the same as in the last Proposition.*

(184.) Retaining every thing as in Art. 181. we have

$$\dot{F} = \frac{\frac{1}{n}\dot{v} - \frac{a-ex}{n-nx^2}v\dot{v}}{1+2av+v^2} + \frac{\frac{1}{n}\dot{v} - \frac{b-fx}{n-nx^2}v\dot{v}}{1-2bv+v^2} + \&c.$$

the fluent of each of which quantities is found as in Art. 139.

PROP. CXXI.

Let $\dot{F} = \frac{\dot{v}}{1 + v^n}$, n *being an* even *number, to find* F.

(185.) By Art. 183. $\dot{F} = \frac{\frac{2}{n}\dot{v} - \frac{2a}{n}v\dot{v}}{1 - 2av + v^2} + \frac{\frac{2}{n}\dot{v} - \frac{2b}{n}v\dot{v}}{1 - 2bv + v^2}$ + &c., whose fluents are found by Art. 139.

If n be an *odd* number, then $\dot{F} = \frac{\frac{2}{n}\dot{v} - \frac{2a}{n}v\dot{v}}{1 - 2av + v^2}$ + &c. + $\frac{\frac{1}{n}\dot{v}}{1 + v}$, whose fluents are found by Art. 139. and 45.

PROP. CXXII.

Let $\dot{F} = \frac{\dot{v}}{1 - v^n}$, n *being an* even *number, to find* F.

(186.) By Art. 183. $\dot{F} = \frac{\frac{2}{n}\dot{v} - \frac{2a}{n}v\dot{v}}{1 - 2av + v^2}$ + &c. + $\frac{\frac{1}{n}\dot{v}}{1 - v}$ + $\frac{\frac{1}{n}\dot{v}}{1 + v}$, whose fluents are found by Art. 139. and 45.

If n be an *odd* number, we have $\dot{F} = \frac{\frac{2}{n}\dot{v} - \frac{2a}{n}v\dot{v}}{1 - 2av + v^2}$ + &c. + $\frac{\frac{1}{n}\dot{v}}{1 + v}$, whose fluents are found by Art. 139. and 45.

PROP. CXXIII.

Let $\dot{F} = \frac{x^r \dot{x}}{1 \pm x^n}$, *to find* F.

Resolve $\frac{1}{1 \pm x^n}$ into $\frac{A - Bx}{1 - 2ax + x^2}$ + &c. + $\frac{P}{1 - x}$ + $\frac{P}{1 + x}$, the two last terms to be taken in according to circumstances ; hence, $\dot{F} = \frac{Ax^r \dot{x} - Bx^{r+1} \dot{x}}{1 - 2ax + x^2}$ + &c. + $\frac{Px^r \dot{x}}{1 - x} + \frac{Px^r \dot{x}}{1 + x}$, and F is found by Prop. 71.

PROP. CXXIV.

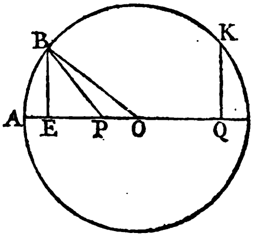
Let $\dot{F} = \frac{\dot{x}}{\sqrt[n]{ax^r + x^n}}$, *to find* F.

Put $ax^r + x^n = x^n z^n$, then $z^n - 1 = ax^{r-n}$; hence, $\frac{nz^{n-1}\dot{z}}{z^n - 1} = (r - n) \times \frac{\dot{x}}{x}$, and $\dot{F} = \frac{\dot{x}}{xz} = \frac{n}{p - n} \times \frac{z^{n-2}\dot{z}}{z^n - 1}$, and F is found by Prop. 72.

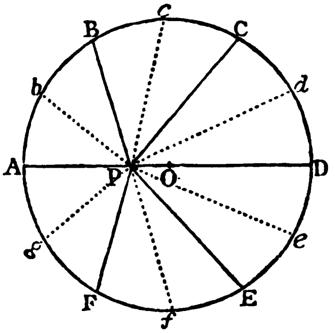
PROP. CXXV.

To demonstrate COTES's *properties of the circle.*

(187.) Retaining every thing as in Art. 179, we have $v^{2n} - 2xv^n + 1^{2n} = 0$, of which, $v^2 - 2yv + 1^2 = 0$



is a quadratic divisor. Assume any point P, and draw PB, and put $v=PO$; then $BO^2=BP^2+PO^2+2PO \times PE$; that is, $1^2=BP^2+v^2+2v\times(y-v)=BP^2-v^2+2yv$; hence, $BP^2=v^2-2yv+1^2$. Also, y is the cosine of $\frac{AK}{n}$, $\frac{360°+AK}{n}$, $\frac{2\times 360°+AK}{n}$, &c. whose cosines are a, b, c, &c. and $(v^2-2av+1^2)\times(v^2-2bv+1^2)\times$ &c.$=v^{2n}-2xv^2+1^{2n}$. Now let AK be the whole circumference C, then the above arcs are $\frac{C}{n}$, $\frac{2C}{n}$, $\frac{3C}{n}$, &c. or



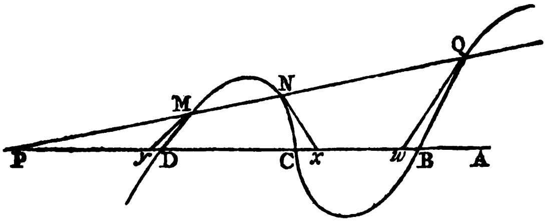
the $\frac{1}{n}$, $\frac{2}{n}$, $\frac{3}{n}$, &c. parts of C; that is, if the whole circumference C be divided from A into $n=$ parts at B, C, D, &c. then the cosines of the arcs AB, AC, AD, &c. are a, b, c, &c. and $x=1$; hence, from what we have already proved, $PB^2=v^2-2av+1^2$, $PC^2=v^2-2bv+1^2$, $PD^2=v^2-2cv+1^2$, &c. consequently $PB^2\times PC^2\times PD^2\times$ &c.$=v^{2n}-2v^n+1^{2n}$; hence, by taking the square root, we get $PB\times PC\times PD\times$ &c. $=v^n-1^n$, or $1^n-v^n=PO^n-AO^n$, or AO^n-PO^n, according as PO or AO is the greater, or according as P is without or within the circle, for every thing holds the same whether P be within or without. This is one of the properties of the circle.

(188.) Let these divisions be again divided into two equal parts at b, c, d, &c. then the whole circumference will be divided into $2n$ equal parts, and therefore from what is already proved, $Pb \times PB \times Pc \times PC \times Pd \times PD \times$ &c. $= AO^{2n} - PO^{2n}$, taking P within, for instance; divide this by the above equation, and we get, $\frac{Pb \times PB \times Pc \times PC \times Pd \times PD \times \&c.}{PB \times PC \times PD \times \&c.} = \frac{AO^{2n} - PO^{2n}}{AO^{n} - PO^{n}}$; that is, $Pb \times Pc \times Pd \times$ &c. $= AO^{n} + PO^{n}$, which is the other property.

Prop. CXXVI.

Let AP *be the abscissa of any curve,* PMNQ *an ordinate revolving about any fixed point* P, *and cutting the curve in as many points as it has dimensions ; and draw the tangents* My, Nx, Qw, &c. *then will* $\frac{1}{\text{Py}} + \frac{1}{\text{Px}} + \frac{1}{\text{Pw}}$ + &c. (*the sum of the reciprocal subtangents*) *be a constant quantity.*

(189.) Let the equation of the curve be $y^n - (a' + b'x) \times y^{n-1}$ + &c. $+ px^n - qx^{n-1}$ + &c. $= 0$; and corresponding to AP the abscissa (x), let a, b, c, &c. be the values of y; then (Wood's *Algebra*, Art. 271.) $a \times b \times c \times$ &c. $= px^n - qx^{n-1}$ + &c. take the fluxion of each side,



and $\dot{a}bc$ &c. $+ \dot{b}ac$ &c. $+ \dot{c}ab$ &c. + &c. $= npx^{n-1}\dot{x} - (n-1) \times qx^{n-2}\dot{x}$ + &c. divide this latter equation by the former, and we have $\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}$ + &c. $=$

$\frac{npx^{n-1}\dot{x} - (n-1) \times qx^{n-2}\dot{x} + \&c.}{px^n - qx^{n-1} + \&c.}$; hence, $\frac{\dot{a}}{a\dot{x}} + \frac{\dot{b}}{b\dot{x}} + \frac{\dot{c}}{c\dot{x}} + \&c. = \frac{npx^{n-1} - (n-1) \times qx^{n-2} + \&c.}{px^n - qx^{n-1} + \&c.}$; but (Art. 23.) $\frac{\dot{a}}{a\dot{x}}$, $\frac{\dot{b}}{b\dot{x}}$, $\frac{\dot{c}}{c\dot{x}}$, &c. are the reciprocals of the subtangents Py, Px, Pw, &c.; hence, (dividing the numerator and denominator on the right hand side of the equation by p, which will not alter it's value)

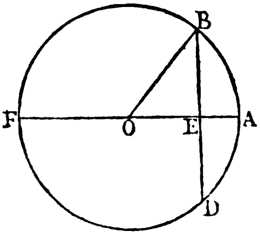
$$\frac{nx^{n-1} - (n-1) \times \frac{q}{p}x^{n-2} + \&c.}{x^n - \frac{p}{q}x^{n-1} + \&c.} = \frac{1}{Py} + \frac{1}{Px} + \frac{1}{Pw} + \&c.$$

But (*Algebra*, Art. 523.) the roots of the equation $x^n - \frac{q}{p}x^{n-1} + \&c. = 0$ are AB, AC, AD, &c. whatever be the angle at P; hence, (*Algebra*, Art. 271.), the coefficients of $x^n - \frac{q}{p}x^{n-1} + \&c.$ are constant; and if P be assumed a fixed point, x is invariable; hence, x^{n-2} $\frac{q}{p}x^{n-2} + \&c.$ is constant, and $nx^{n-1} - (n-1) . \frac{q}{p} x^{n-2} + \&c.$ is constant; therefore the sum of the reciprocal subtangents is a constant quantity.

Prop. CXXVII.

Given the arc of a circle; to find it's sine and cosine.

(190.) Put the radius $OA = r$, the arc $AB = z$, it's sine $BE = x$, cosine $OE = y$, and produce BE to D; then (Art. 46.) $\dot{z} : -\dot{y} :: r : x = \frac{-r\dot{y}}{\dot{z}}$. Now corresponding to the same value OE of y, z may be either AB or AD; but the arc beginning at A, if we consider AB as positive, AD will be negative, therefore every positive value of z* has a negative value equal to it; hence, by the note, if we assume y in a series of the powers of z, only the even powers of z will enter.



Assume therefore $y = r + az^2 + bz^4 + cz^6 + \&c.$ the first term being r, because when $z = 0$, $y = r$; hence, $\dot{y} = 2az\dot{z} + 4bz^3\dot{z} + 6cz^5\dot{z} + \&c.$ therefore $x \left(= \frac{-r\dot{y}}{\dot{z}}\right) = -2raz - 4rbz^3 - 6rcz^5 - \&c.$ and $\dot{x} = -2ra\dot{z} - 3.4rbz^2\dot{z} - 5.6rcz^4\dot{z} - \&c.$ But (Art. 46.) $\dot{z} : \dot{x} :: r : y$; hence, $y\dot{z} = r\dot{x}$, and $y\dot{z} - r\dot{x} = 0$; now in this equation, instead of y and $\dot{x}$ substitute their values above found, and we have

$$\left.\begin{array}{l} r\dot{z} + \qquad az^2\dot{z} + \qquad bz^4\dot{z} + \&c. \\ 2r^2a\dot{z} + 3.4r^2bz^2\dot{z} + 5.6r^2cz^4\dot{z} + \&c. \end{array}\right\} = 0;$$

hence, (Art. 110.) $2r^2a + r = 0$, $3.4r^2b + a = 0$, $5.6r^2c + b = 0$, &c. consequently $a = \frac{-1}{2r}$; $b = \frac{-a}{3.4r^2} = \frac{1}{2.3.4r^3}$; $c = \frac{-b}{5.6r^2} = \frac{-1}{2.3.4.5.6r^5}$; &c. hence, $y =$

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* If every positive value of z have a negative value equal to it, the equation whose roots are those values of z, will have only the *even* powers of z; for if $z = a$, $z = -a$, then $z - a = 0$, $z + a = 0$, and consequently the quadratic from these two will be $z^2 - a^2 = 0$; and as every such pair of roots will form a similar quadratic, it is manifest, that the equation formed by the multiplication of these quadratics, will contain only the even power of z.

$r - \frac{z^2}{2r} + \frac{z^4}{2.3.4r^3} - \frac{z^6}{2.3.4.5.6r^5} + \text{\&c.}$ Also, $-2ra = 1$; $-4rb = \frac{-1}{2.3r^2}$; $-6rc = \frac{1}{2.3.4.5r^4}$; &c. hence, $x = z - \frac{z^3}{2.3r^2} + \frac{z^5}{2.3.4.5r^4} - \text{\&c.}$

Prop. CXXVIII.

To find the sum of the series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \text{\&c.}$ ad infinitum.

(191.) Put the radius $AO = 1$, $EB = x$, $AB = z$; then (Art. 190.) $x = z - \frac{z^3}{2.3} + \frac{z^5}{2.3.4.5} - \text{\&c.}$ Let $x = 0$, and then $z - \frac{z^3}{2.3} + \frac{z^5}{2.3.4.5} - \text{\&c.} = 0$, or $1 - \frac{z^2}{2.3} + \frac{z^4}{2.3.4.5} - \text{\&c.} = 0$, the former equation containing one root $= 0$, it being divisible by z, or $z - 0$, (*Alg.* Art. 268.) which is taken away by dividing by z. But if $c =$ the semi-circumference of the circle, the other values of z, corresponding to $x = 0$, will be $1c$, $2c$, $3c$, &c. *ad infinitum*, and by taking the arcs in a contrary direction, they will be $-1c$, $-2c$, $-3c$, &c. *ad infinitum* (*Algebra*, 471.); hence, these values of z are the roots of the equation $1 - \frac{z^2}{2.3} + \frac{z^4}{2.3.4.5} - \text{\&c.} = 0$. Put $z = \frac{1}{y}$, and the equation becomes $1 - \frac{1}{2.3.y^2} + \frac{1}{2.3.4.5.y^4} - \text{\&c.} = 0$; multiply it by y^n, and it becomes $y^n - \frac{y^{n-2}}{1.3}$

$+\frac{y^{n-4}}{2.3.4.5} -$ &c. $=0$, which equation contains n roots $=0$, the other roots remaining the same. But as $y=\frac{1}{x}$, the values of y are $\frac{1}{1c}, \frac{1}{2c}, \frac{1}{3c}$, &c. and $-\frac{1}{1c}, -\frac{1}{2c}, -\frac{1}{3c}$, &c. *ad inf.* Now (*Alg.* Art. 352.) the sum of the squares of the roots of the last equation is $\frac{1}{3}$; and the squares of the positive values of y being the same as the square of the negative values, we have $\frac{2}{1^2 c^2} + \frac{2}{2^2 c^2} + \frac{2}{3^2 c^2} +$ *ad inf.* $= \frac{1}{3}$, consequently $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} +$ &c. *ad inf.* $= \frac{c^2}{6}$.

Cor. 1. In like manner we may find the sum of any of the *even* powers of the reciprocals of the natural numbers, by assuming the sum equal to it's value given by the same Art. in the *Algebra*. For instance, the sum of the fourth powers of the roots of the equation is $\frac{1}{45}$; hence, $\frac{2}{1^4 c^4} + \frac{2}{2^4 c^4} + \frac{2}{3^4 c^4} +$ &c. $= \frac{1}{45}$, consequently $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} +$ &c. $= \frac{c^4}{90}$.

The sum of the reciprocals of the *odd* powers cannot be found by this method, because the *odd* powers of the *negative* roots destroy those of the *positive*.

Cor. 2. By transposition, $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} +$ &c. $= \frac{c^2}{6} - \frac{1}{2^2} - \frac{1}{4^2} -$ &c. $= \frac{c^2}{6} - \frac{1}{2^2} \times \left(\frac{1}{1^2} + \frac{1}{2^2} + \text{\&c.}\right) = \frac{c^2}{6} -$

$\frac{1}{2^2} \times \frac{c^2}{6} = \frac{c^2}{8}$. And in like maner, we may find the sum of the reciprocals of all the even powers of 1, 3, 5, &c.

Prop. CXXIX.

Supposing the force of gravity to vary as the n^{th} *power of the distance from the centre of the earth, and the compressive force of the air to vary as it's density; to find the density of the air at any altitude above the surface of the earth.*

(192.) Let the radius of the earth = 1, x = the distance of any point above the earth's surface from the centre, v = the density of the air at that point, the density at the surface being unity; h = the altitude of an homogeneous atmosphere. Now it appears by experiment, that the compressive force of the air in the atmosphere, varies as it's density; consequently the fluxion of the compressive force must be to the fluxion of the density, as the compressive force is to the density, and this ratio is the same at all altitudes. Now at any distance x from the earth's centre, the fluxion of the compressive force must be in proportion to the force of gravity, the density, and the fluxion of the altitude; hence, $x^n v\dot{x}$ has a constant ratio to $-\dot{v}$, writing the latter fluxion with the sign $-$ (Art. 16.), because v decrease as x increases; and according to this representation of the compressive force, h will represent the compressive force at the surface; for at the surface, $v = 1$, $x^n = 1$, and this force is supposed to be the same for the height h; therefore for the homogeneous atmosphere, where v is constant and $= 1$, the fluxion $x^n v\dot{x}$ becomes $\dot{x}$, and the fluent is x, and it becomes h for the whole altitude. Hence, $h : 1 :: x^n v\dot{x} : -\dot{v}$, therefore $x^n\dot{x} = -h \times \frac{\dot{v}}{v}$, and $\frac{x^{n+1}}{n+1} = -h \times \text{h. l. } v + C$; but when $x = 1$,

$v = 1$, and this equation becomes $\frac{1}{n+1} = C$; hence, the correct fluent is $\frac{x^{n+1}}{n+1} = -h \times \text{h.l.}\ v + \frac{1}{n+1}$, consequently $\frac{1-x^{n+1}}{n+1} = h \times \text{h.l.}\ v$, an equation expressing the relation between the altitude and density.

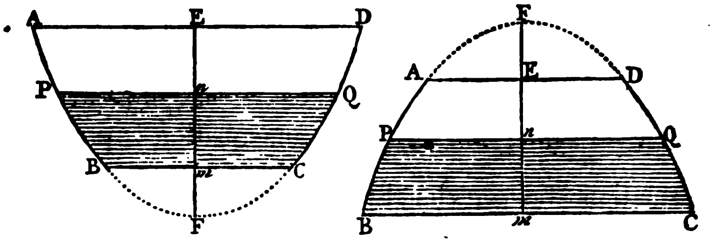
Cor. 1. If we suppose the force to vary inversely as the square of the distance, n becomes -2; hence, $\frac{1}{x} - 1 = h \times \text{h.l.}\ v$; if therefore x increase in musical progression, $\frac{1}{x}$ will decrease in arithmetic progression, and consequently the h. l. v will decrease in arithmetic progression.

Cor. 2. If the force of gravity be supposed constant, $n = 0$; hence, $1 - x = h \times \text{h.l.}\ v$; and if x increase in arithmetic progression, then $1 - x$ will decrease in arithmetic progression, consequently the h. l. v will decrease in arithmetic progression.

Prop. CXXX.

To find the time in which a vessel ABCD *filled with a fluid, will empty itself through a very small orifice* m *at the bottom.*

(193.) Put $a = 32\frac{1}{6}$ feet $= 386$ inches, $x = mn$ the depth of the fluid at any point of time, $z =$ the area of the surface PQ of the fluid, $m =$ the area of the orifice, $t =$ the time in which the surface of the fluid descends from PQ to BC. Now it appears by experiment, that the velocity of the fluid at the orifice, is that which a body acquires in falling down $\frac{1}{2}\,x$, supposing the orifice to be very small compared with



the surface of the fluid; hence, by *Mechanics*, $\sqrt{\frac{1}{2}a} : \sqrt{\frac{1}{2}x} :: a : \sqrt{ax}$ = the velocity (per second) at the orifice; and by the Principles of *Hydrostatics*, $z : m :: \sqrt{ax} : \frac{m}{z} \times \sqrt{ax}$ the velocity with which the surface descends; hence, (Art. 82.) $\dot{t} = \frac{\dot{x}}{\frac{m}{z} \times \sqrt{ax}} = \frac{z\dot{x}}{m\sqrt{ax}}$, the fluent of which, corrected when necessary, gives t.

EXAMPLES.

Ex. 1. *Let the vessel be a cylinder or prism.*

Put $h = Em$ it's altitude. In this case z is constant, and $\dot{t} = \frac{z\dot{x}}{m\sqrt{ax}} = \frac{z}{m\sqrt{a}} \times x^{-\frac{1}{2}}\dot{x}$, whose fluent is $t = \frac{2zx^{\frac{1}{2}}}{m\sqrt{a}} = \frac{2z}{m} \times \sqrt{\frac{x}{a}}$, which wants no correction; and when $x=h$, $t = \frac{2z}{m} \times \sqrt{\frac{h}{a}}$, the time of emptying.

Ex. 2. *Let* ABCD *be the frustrum of a cone.*

Put $Fm = c$, $mB = d$, $Em = e$, $p = 3{,}14159$, &c. then $Fn = c \pm x$, the sign + or − being taken, according as the less or greater end is downwards; and (*FA*, *FD* being now right lines) by similar triangles, $c : d :: c \pm x : Pn = \frac{d}{c} \times (c \pm x)$; hence, $z = \frac{pd^2}{c^2} \times (c \pm x)^2$; consequently $\dot{t} = \frac{pd^2}{mc^2\sqrt{a}} \times x^{-\frac{1}{2}} \times (c \pm x)^2 \times \dot{x} = \frac{pd^2}{mc^2\sqrt{a}} \times (c^2 x^{-\frac{1}{2}}\dot{x} \pm 2cx^{\frac{1}{2}}\dot{x} + x^{\frac{3}{2}}\dot{x})$, and $t = \frac{pd^2}{mc^2\sqrt{a}} \times (2c^2 x^{\frac{1}{2}} \pm \frac{4}{3} c x^{\frac{3}{2}} + \frac{2}{5} x^{\frac{5}{2}})$, which requires no correction; and when $x = e$, $t = \frac{pd^2}{mc^2\sqrt{a}} \times (2c^2 e^{\frac{1}{2}} \pm \frac{4}{3} c e^{\frac{3}{2}} + \frac{2}{5} e^{\frac{5}{2}}$, the whole time of emptying.

If the orifice be a circle whose radius $= r$, then $m = pr^2$; consequently $t = \frac{d^2}{r^2 c^2 \sqrt{a}} \times (2c^2 e^{\frac{1}{2}} \pm \frac{4}{3} c e^{\frac{3}{2}} + \frac{2}{5} e^{\frac{5}{2}})$.

Cor. If the base be downwards, and we take the whole cone, then $c = e$; hence, $t = \frac{d^2}{r^2 c^2 \sqrt{a}} \times \frac{16}{15} c^{\frac{5}{2}} = \frac{16 d^2 \sqrt{c}}{15 r^2 \sqrt{a}}$, the whole time of emptying.

If the vertex be downwards, and the orifice be so small that we may consider *Em* as equal to *EF*, then $c = 0$, $d = 0$; but because c is always to d as $FE : EA$, $\therefore$ when c and d vanish, we may consider $\frac{d^2}{c^2} = \frac{EA^2}{FE^2}$; hence, $t = \frac{EA^2}{FE^2 \times r^2 \sqrt{a}} \times \frac{2}{5} e^{\frac{5}{2}} = \frac{2 EA^2 \times \sqrt{FE}}{5 r^2 \sqrt{a}}$ the whole time of emptying.

Ex. 3. *Let* BFC *be an hemisphere standing on it's base.*

Put the radius $mB = mF = r$; then $Pn^2 = r^2 - x^2$, and $z = p \times (r^2 - x^2)$; hence, $\dot{t} = \frac{p \times (r^2 - x^2) \times \dot{x}}{m\sqrt{ax}} = \frac{p}{m\sqrt{a}} \times (r^2 x^{-\frac{1}{2}}\dot{x} - x^{\frac{3}{2}}\dot{x})$, whose fluent is $t = \frac{p}{m\sqrt{a}} \times (2r^2 x^{\frac{1}{2}} - \frac{2}{5}x^{\frac{5}{2}})$, which wants no correction; and when $x = r$, $t = \frac{8p}{5m\sqrt{a}} \times r^{\frac{5}{2}}$, the whole time of emptying.

If the orifice be a circle whose radius is w, then $m = pw^2$; hence, $t = \frac{8r^{\frac{5}{2}}}{5w^2\sqrt{a}}$.

If the hemisphere stand on it's vertex, $Pn^2 = 2rx - x^2$; hence, $z = p \times (2rx - x^2)$, consequently $\dot{t} = \frac{p}{m\sqrt{a}} \times (2rx^{\frac{1}{2}}\dot{x} - x^{\frac{3}{2}}\dot{x})$, whose fluent is $t = \frac{p}{m\sqrt{a}} \times (\frac{4}{3}rx^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}})$, which requires no correction; and when $x = r$, $t = \frac{14pr^{\frac{5}{2}}}{15m\sqrt{a}} = \frac{14r^{\frac{5}{2}}}{15w^2\sqrt{r}}$, the whole time of emptying.

Ex. 4. *Let* BCF *be a* paraboloid *standing on it's base.*

Put it's parameter $= r$, it's altitude $Fm = e$, then $r \times (e - x) = Pn^2$, and $pr \times (e - x) = z$; hence, $\dot{t} = \frac{pr}{m\sqrt{a}} \times (ex^{-\frac{1}{2}}\dot{x} - x^{\frac{1}{2}}\dot{x})$, whose fluent is $t = \frac{pr}{m\sqrt{a}} \times$

$(2ex^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}})$, which requires no correction; and when $x = e$, $t = \frac{4pre^{\frac{3}{2}}}{3m\sqrt{a}} = \frac{4re^{\frac{3}{2}}}{3w^2\sqrt{a}}$, the whole time of emptying.

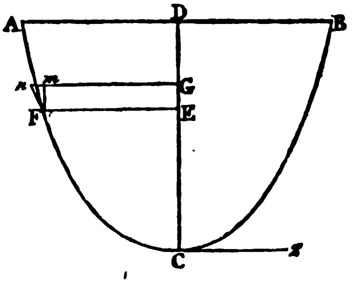
If the paraboloid stand on it's vertex, $Pn^2 = rx$; hence, $z = prx$; consequently $\dot{t} = \frac{prx^{\frac{1}{2}}\dot{x}}{m\sqrt{a}}$, and $t = \frac{2prx^{\frac{3}{2}}}{3m\sqrt{a}}$, which wants no correction; and when $x = e$, $t = \frac{2pre^{\frac{3}{2}}}{3m\sqrt{a}} = \frac{2re^{\frac{3}{2}}}{3w^2\sqrt{a}}$, the whole time of emptying.

In like manner, whatever be the form of the vessel, we may find the time of emptying, substituting into the value of $\dot{t}$, the quantity z expressed in terms of x, and then taking the fluent.

Prop. CXXXI.

If a perfectly flexible chain ACB *of uniform density and thickness, be hung upon two pins at* A *and* B; *to find the curve into which it will form itself.*

(194.) Let *C* be the lowest point, draw the axis *CD* perpendicular to the horizon; draw also *EF*, *Gn* perpendicular to *CD*; *Fn* a tangent at *F*, and *Fm* perpendicular to *FE*. Now assuming any part *CF* of the chain, we may consider it as if it were perfectly rigid; for conceive *CF* to become perfectly rigid, and it is manifest that no alteration whatever can take place; for the gravity of the chain gives *CF* a certain situation; and if we make that part to become inflexible, we add no new force; we only suppose a cohesion to take place between the constituent particles whilst they are so disposed. Considering therefore *CF* as a perfectly



inflexible body, it is kept at rest by three forces; at *C* by the action of the part *BC* of the chain in the direction *Cz* of the tangent at *C*; at *F* by the action of the part *FA* of the chain in the direction *Fn* of the tangent at *F*; and by it's gravity in a direction parallel to *EC*; but * *Cz* is parallel to *mn*, and *CE* to *mF*; hence, these three forces act parallel to the three sides of the triangle *Emn*, and consequently will be respectively proportional to them, the body *FC* being at rest. Put $CE = x$, $EF = y$, $CF = z$, then (Art. 23. and 27.) $Fm = \dot{x}$, $mn = \dot{y}$, $Fn = \dot{z}$. Now the chain being of uniform density and thickness, the gravity of any part *CF* will be in proportion to it's length z; also, let $a =$ the tension of the chain *BC* at *C* acting in the direction *Cz*, (a constant quantity, it not varying by changing the point *F*), represented by a length of chain whose weight at *C* acting in the direction *Cz* would keep the chain *AC* in it's position. Hence, $a : z :: \dot{y} : \dot{x}$, $\therefore a\dot{x} = z\dot{y}$; but $\dot{z}^2 = \dot{x}^2 + \dot{y}^2 = \dot{x}^2 + \frac{a^2\dot{x}^2}{z^2}$, therefore $z^2\dot{z}^2 = z^2\dot{x}^2 + a^2\dot{x}^2$, consequently $\dot{x} = \frac{z\dot{z}}{\sqrt{a^2+z^2}}$, whose fluent (Art. 39.)

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* As by *Mechanics*, these three forces must be directed to one point, if the two tangents *nF*, *zC* be produced to meet, the intersection must be in the line of direction passing through the centre of gravity of *FC*.

is $x = \sqrt{a^2 + z^2} + C$; but when $x = 0$, then $z = 0$; hence, the equation becomes $0 = a + C$, and $C = -a$; therefore the correct fluent is $x = \sqrt{a^2 + z^2} - a$, and by transposing a and squaring both sides, $x^2 + 2ax = z^2$, the equation of the curve. This curve is called the *Catenary*.

From the equation $z^2 = x^2 + 2ax$, we get $z\dot{z} = x\dot{x} + a\dot{x} = (a+x) \times \dot{x}$, and $z^2\dot{z}^2 = (a+x)^2 \times \dot{x}^2$, or $\left((a+x)^2 - a^2\right) \times \dot{z}^2 = (a+x)^2 \times (\dot{z}^2 - \dot{y}^2)$; hence, $a\dot{z} = (a+x) \times \dot{y}$.

Further, as $\dot{y} = \frac{a\dot{x}}{z} = \frac{a\dot{x}}{\sqrt{x^2 + 2ax}}$, we have $y = a \times$ h. l. $\frac{a + x + \sqrt{x^2 + 2ax}}{a} = a \times$ h. l. $\frac{z + \sqrt{a^2 + z^2}}{a}$. Hence, if c = the number whose h. l. $= 1$, then, (Art. 112.) $c^{\frac{y}{a}} = \frac{a + x + \sqrt{x^2 + 2ax}}{a} = \frac{z + \sqrt{a^2 + z^2}}{a}$.

If *B* be any where between *A* and *C*, so that *B* becomes the lowest point, we must consider the curve as produced to *C*, and then the same equation obtains for the same abscissa and ordinate.

Prop. CXXXII.

If the chain ACB *be of uniform thickness; to find the law of weight and density, so that it may form itself into any given curve.*

(195.) Let w = the weight of any part *CF*, d = the density at *F*; then by the last Proposition, $a : w :: \dot{y} : \dot{x}$; therefore $w = a \times \frac{\dot{x}}{\dot{y}}$. Now $\dot{w} = d\dot{z}$; hence, $d = \frac{\dot{w}}{\dot{z}}$.

But $w = a \times \frac{\dot{x}}{\dot{y}}$, and if $\dot{y}$ be made constant, $\dot{w} = a \times \frac{\ddot{x}}{\dot{y}}$; hence, $d = \frac{a\ddot{x}}{\dot{y}\dot{z}}$, which gives the law of density.

EXAMPLES.

Ex. 1. *Let the curve be a* circle *whose radius is* r.

Here, $\dot{x} : \dot{y} :: y : r - x$; therefore $w \left(= a \times \frac{\dot{x}}{\dot{y}}\right) = a \times \frac{y}{r-x} = a \times$ tan. of CF; the weight therefore of any part CF varies as the tangent of CF. Now, $y^2 = 2rx - x^2$, and $y\dot{y} = r\dot{x} - x\dot{x}$, and (making $\dot{y}$ constant) $\dot{y}^2 = r\ddot{x} - x\ddot{x} - \dot{x}^2$, therefore $\ddot{x} = \frac{\dot{y}^2 + \dot{x}^2}{r-x} = \frac{\dot{z}^2}{r-x} =$ (because $r : y :: \dot{z} : \dot{x}$) $\frac{r^2\dot{x}^2}{y^2 \times (r-x)}$; also, $\dot{y} = \frac{(r-x) \times \dot{x}}{y}$, and $\dot{z} = \frac{r\dot{x}}{y}$; hence, $d \left(= \frac{a\ddot{x}}{\dot{y}\dot{z}}\right) = \frac{ar^2\dot{x}^2}{y^2 \times (r-x)} \times \frac{y}{(r-x) \times \dot{x}} \times \frac{y}{r\dot{x}} = \frac{ar}{(r-x)^2}$. The density therefore varies inversely as the square of the cosine of CF. If therefore the arc be a semicircumference, the density at the highest point is infinite.

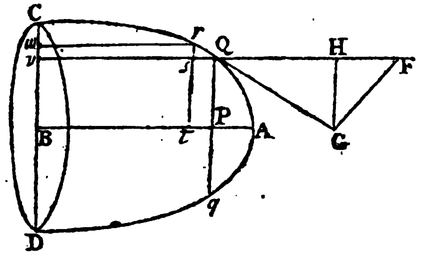
Ex. 2. *Let the curve be a* parabola.

Here, $px = y^2$; therefore, $\dot{x} = \frac{2y\dot{y}}{p}$; hence, $w \left(= a \times \frac{\dot{x}}{\dot{y}}\right) = \frac{2ay}{p}$; therefore the weight of any part CF varies as the ordinate FE. Also, (if $\dot{y}$ be constant) $\ddot{x} = \frac{2\dot{y}^2}{p}$; but (Art. 54. Ex. 3.) $\dot{z} = \frac{\overline{y^2 + c^2}|^{\frac{1}{2}} \times \dot{y}}{c}$, putting $c = \frac{1}{2}p$; hence, $d \left(= \frac{a\ddot{x}}{\dot{y}\dot{z}}\right) = \frac{a}{\sqrt{y^2 + c^2}}$. The density therefore varies inversely as $\sqrt{y^2 + c^2}$, or inversely as the *normal* (Art. 24. Ex.).

Prop. CXXXIII.

Let CAD *be a plane figure, or a solid generated by it's revolution about it's axis, moving in a fluid in the direction of it's axis* BA; *to find the proportion of the resistance on the curve line* CAD, *or of the surface of the solid, to the resistance on the base* CD.

(196.) Draw $FQsv$ and wr parallel to AB, rst,



QPq perpendicular to AB; then if $AP = x$, $PQ = y$, $QA = z$, it appears from Arts. 23. and 27. that ultimately, by bringing r up to Q, $Qs = \dot{x}$, $sr = \dot{y}$, $Qr = \dot{z}$. Draw the tangent QG, and let fall the perpendicular FG upon it, and also GH upon FQ. Now let FQ represent the force of one particle of the fluid, then if that particle struck the base at v, it's whole force would act to oppose the motion, because it acts perpendicularly to the base, and therefore no part of it's force is lost; but striking the curve at Q obliquely, if the force FQ be resolved into GQ and FG, then GQ is here supposed to be lost by the obliquity of the stroke, and FG to be the only effective part; but this not being opposite to the motion of the body, we must resolve it into FH and HG, and then FH is that part which opposes the motion of the body, and HG is destroyed by an equal and opposite force of a particle acting at q. Hence, the force of a particle at v: force at Q :: $FQ : FH$ :: (because $FQ : FG :: FG : FH$) $FQ^2 : FG^2$ :: (by sim. trian.) $\dot{x}^2 : \dot{y}^2$. Now the quantity of fluid striking Qr and vw is the same, and in proportion to sr or $\dot{y}$. Hence, if we consider it as a *plane* figure, as the whole force is as the number of particles × force of each, we have the force against vw : force against Qr :: $\dot{y} : \frac{\dot{y}^3}{\dot{z}^2} = \frac{\dot{y}^3}{\dot{x}^2 + \dot{y}^2} = \frac{\dot{y}}{1 + \frac{\dot{x}^2}{\dot{y}^2}}$; hence, the whole resistance on the base : that on the curve :: the fluent of $\dot{y}$, or y : fluent (F) of $\frac{\dot{y}}{1 + \frac{\dot{x}^2}{\dot{y}^2}}$.

For a *solid*, the number of particles striking the area generated by vw will be as vw × circum. described by v, or as $vw \times y$, or as $y\dot{y}$; hence, for the same reason, the resistance on the base : that on the surface :: the flu. of $y\dot{y}$, or $\frac{1}{2}y^2$, : flu. (F) of $\frac{y\dot{y}}{1 + \frac{\dot{x}^2}{\dot{y}^2}}$.

EXAMPLES.

Ex. 1. *Let* ACD *be an* isosceles triangle.

Here the plane is a triangle, and $\dot{x} : \dot{y} :: x : y :: a$ $(AB) : b\,(BC)$, $\therefore \frac{\dot{x}^2}{\dot{y}^2} = \frac{a^2}{b^2}$, hence, the resistances are as y : flu. $\frac{\dot{y}}{1 + \frac{a^2}{b^2}} :: y : \frac{y}{1 + \frac{a^2}{b^2}} :: b^2 + a^2 : b^2 :: AC^2 : BC^2$. The same is true for the *cone*, or for any *prismatic* solid.

Ex. 3. *Let* CAD *be a* semicircle.

Put $AB = r$, then $y^2 = 2rx - x^2$; hence, $\dot{x} = \frac{y\dot{y}}{r - x} =$

$\frac{y\dot{y}}{\sqrt{r^2-y^2}}$, and $\frac{\dot{x}^2}{\dot{y}^2}=\frac{y^2}{r^2-y^2}$; $\therefore \dot{F}=\frac{\dot{y}}{1+\frac{y^2}{r^2-y^2}}=\frac{r^2\dot{y}-y^2\dot{y}}{r^2}$,

and $F=y-\frac{y^3}{3r^2}$; hence, the resistances are as $y : y-\frac{y^3}{3r^2}$; which, when $y=r$, is as 3 : 2.

Ex. 3. *Let* CAD *be an* hemisphere.

Here $\dot{F}=\frac{y\dot{y}}{1+\frac{y^2}{r^2-y^2}}=\frac{r^2y\dot{y}-y^3\dot{y}}{r^2}$, and $F=\frac{1}{2}y^2-\frac{y^4}{4r^2}$;

hence, the resistances are as $\frac{1}{2}y^2 : \frac{1}{2}y^2-\frac{y^4}{4r^2}$, which, when $y=r$, is as 2 : 1.

Ex. 4. *Let the solid* CAD *be generated by a* cycloid AC *revolving about* AB, BC *being the axis of the cycloid.*

If $a=BC$, then $y=z-\frac{z^2}{4a}$ by the nature of the curve;

hence, $\dot{y}=\dot{z}-\frac{z\dot{z}}{2a}$, $\therefore \dot{F}=\frac{y\dot{y}\times\dot{y}^2}{\dot{z}^2}=\frac{y\dot{y}\times(2a-z^2)}{4a^2}=\frac{y\dot{y}\times(a-y)}{a}=y\dot{y}-\frac{y^2\dot{y}}{a}$, and $F=\frac{1}{2}y^2-\frac{y^3}{3a}$; hence, the resistances are as $\frac{1}{2}y^2 : \frac{1}{2}y^2-\frac{y^3}{3a}$, which, when $y=a$, is as 3 : 1.

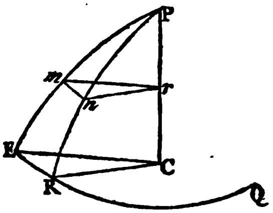
(197.) Considering the body as a *solid*, and the force of a particle on the base as constant, the force of a particle on the surface $\propto\frac{\dot{y}^2}{\dot{z}^2}$, and the area generated by rs being as $y\dot{y}$, the resistance against $Qr\propto\frac{y\dot{y}^3}{\dot{z}^2}$.

ON MERCATOR'S PROJECTION.

PROP. CXXXIV.

If P *be the pole of the earth,* EQ *the equator,* PE, PR, *two meridians,* mn *a small circle parallel to* ER; *then the length of a degree of latitude : the length of a degree of longitude at* m :: *radius : the cosine of the latitude of* m, *supposing the earth to be a sphere.*

(198.) For let PC be the radius of the earth; draw



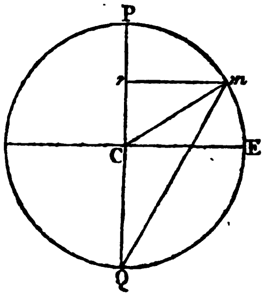
mr, nr perpendicular to it, and join EC, RC. Then mr, nr being parallel to EC, RC respectively, the angle $mrn = ECR$; hence, by similar sectors, $ER : mn :: EC : mr$ the cosine of mE. But when the angle is given, the length of an arc of a degree is in proportion to the radius; also, the length of a degree of the great circle ER, is a degree of latitude; and the length of a degree of mn is a degree of longitude at m; hence, a degree of latitude : a degree of longitude :: radius : the cosine of latitude.

In MERCATOR'S Projection, the sphere is projected upon a plane, and the meridians EP, RP are straight lines parallel to each other; consequently P must be at an infinite distance from the equator EQ. In this case, the arc mn being the same at all latitudes, the length of a degree of longitude is every where the same; to preserve therefore the proper proportion between the degrees of latitude and longitude, the degrees of latitude must increase as you go from the equator, so that they may always be to the degrees of longitude, in the proportion of radius to the cosine of latitude.

Prop. CXXXV.

In this projection, it is required to find the length of an arc of the meridian, corresponding to any given latitude.

(199.) Let *P* be the pole, *E* the equator, *PCQ* a diameter of the earth, *C* the centre; *m* any place on the surface; draw *mr* perpendicular to *PQ*, and join



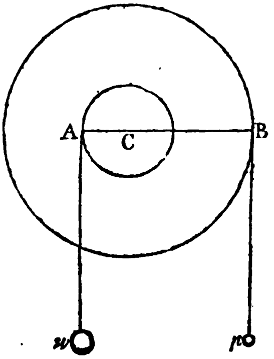
mC, *mQ*. Put $Cm = r$, $Em = x$, Cr (the sine of *Em* the latitude of *m*) $= y$, and the length of *Em* on the projection $= z$, called the *meridional parts*. Then by the last Prop. $\sqrt{r^2 - y^2}$ (cos. of lat.) $: r :: \dot{x} : \dot{z} = \frac{r\dot{x}}{\sqrt{r^2 - y^2}}$; but (Art. 46.) $\dot{x} = \frac{r\dot{y}}{\sqrt{r^2 - y^2}}$; hence, $\dot{z} = \frac{r^2\dot{y}}{r^2 - y^2} = \frac{r}{2} \times \frac{2r\dot{y}}{r^2 - y^2}$, $\therefore$ $z = \frac{r}{2} \times \text{h.l.} \frac{r+y}{r-y} + C$ (Art. 45. Ex. 6.) $= r \times \text{h.l.} \sqrt{\frac{r+y}{r-y}} + C$, by the nature of logarithms. But by Plane Trig. $\sqrt{r^2 - y^2}$ $(mr) : r + y$ $(rQ) :: r$ (rad.) $: \frac{r \times (r+y)}{\sqrt{r^2 - y^2}}$

$=r\sqrt{\frac{r+y}{r-y}}$ the tangent of the angle rmQ = cotangent of rQm = cotan. of $\frac{1}{2}rCm$ = cotan. $\frac{1}{2}$ the complement of lat.; hence, $\sqrt{\frac{r+y}{r-y}} = \frac{\text{cotan. } \frac{1}{2} \text{ comp. lat.}}{r}$; consequently $z = r \times \text{h. l. } \frac{\text{cotan. } \frac{1}{2} \text{ comp. lat.}}{r} + C$; but when $z=0$, cotan. $\frac{1}{2}$ comp. lat. $=r$; hence, $0 = r \times \text{h. l. } \frac{r}{r} + C = r \times \text{h. l. } 1 + C = 0 + C$, $\therefore C = 0$; consequently $z = r \times \text{h. l. } \frac{\text{cotan. } \frac{1}{2} \text{ comp. lat.}}{r} = r \times \text{h. l. cotan. } \frac{1}{2}$ comp. lat. $- r \times \text{h. l. } r$, the length of the meridian Em in the projection.

PROP. CXXXVI.

Given the radii BC, AC *of a wheel and axle, and the weight* p *which draws up* w; *to find* w, *so that the momentum communicated to it in a given time may be a maximum, the wheel and axle being supposed of no weight.*

(200.) Put $BC=b$, $AC=a$; then, by *Mechanics*, the forces with which w and p endeavour to descend, are aw



and bp; hence, the moving force is as $bp-aw$; also, the inertia of each weight is (Art. 60.) as $a^2 \times w$, and $b^2 \times p$; hence, the accelerative force of the lever is as $\frac{bp-aw}{b^2p+a^2w}$; and as the acceleration of any point of a lever must (besides the accelerating force with which the lever itself is made to revolve) be in proportion to the distance of that point from the fulcrum, the accelerative force of the point A, or of w, will be as $\frac{abp-a^2w}{b^2p+a^2w}$, which is as the velocity generated in w in a given time; consequently the momentum of w will be as $\frac{abp-a^2w}{b^2p+a^2w} \times w = \frac{abpw-a^2w^2}{b^2p+a^2w}$ = a maximum, or $\frac{bpw-aw^2}{b^2p+a^2w}$ = a maximum; hence, (Art. 21.) it's fluxion

$$\frac{(bp\dot{w}-2aw\dot{w}) \times (b^2p+a^2w) - a^2\dot{w} \times (bpw-aw^2)}{\overline{b^2p+a^2w}|^2} = 0,$$

or $(bp-2aw) \times (b^2p+a^2w) - a^2 \times (bpw - aw^2) = 0$;

hence, $w = \left(\sqrt{\frac{b^4}{a^4} + \frac{b^3}{a^3}} - \frac{b^2}{a^2}\right) \times p$.

If $a = b$, $w = (\sqrt{2} - 1) \times p$.

Prop. CXXXVII.

Given two weights w *and* p, *and the radius* CA *of the axle, to find the radius* CB *of the wheel, so that* p *may draw up* w *through a given space, in the least time possible.*

(201.) When the space is given, the time varies inversely as the square root of the accelerative force; hence (by the last Art.), the square of the time varies as $\frac{b^2p+a^2w}{abp-a^2w}$ a minimum, where b is variable; put it's fluxion $=0$, and we get $b=\frac{aw}{p}+\frac{\sqrt{a^2w^2+a^2pw}}{p}$.

If $p=w$, $b=a\times(1+\sqrt{2})$.

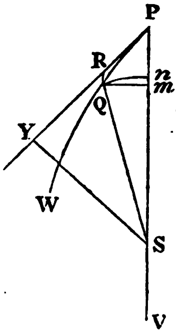
DEFINITION.

The centrifugal force is that force by which a body describing a curve about a centre of force, recedes from that centre in virtue of it's circular motion about it.

PROP. CXXXVIII.

To find the ratio of the centripetal to the centrifugal force of a body describing a curve about a centre of force.

Let S be the centre of force, PQ an indefinitely small part of the curve PW; draw QR parallel to PSV, PV being the chord of curvature; with the centre S describe the circular arc Qn, and draw Qm perpendicular to SP, and SY to the tangent PY. Now PQ is the motion of the body in the indefinitely small time t; resolve the motion PQ into Pm, mQ, and Pm would be the approach of the body towards S in the



time t in virtue of the motion in that direction; but in virtue of the motion mQ in the time t, the body has approached S only by Pn; mn is therefore the space through which the body has receded from S in virtue of it's angular motion about S, and therefore represents the centrifugal force; but QR represents the centripetal force; hence, the centripetal : centrifugal force :: $QR : mn :: \frac{PQ^2}{PV} : \frac{Qn^2}{2PS}$ :: (sim. tri.) $\frac{PS^2}{PV} : \frac{SY^2}{2PS} :: 2PS^3 : SY^2 \times PV$.

Cor. 1. If PW be a circle about the centre S, then $2SP^3 = SY^2 \times PV$, and the two forces become equal. Now $mn = \frac{Qn^2}{2QS} = \frac{Qn^2}{2PS}$; but when the time t is given, mn represents the force, and Qn the velocity. Hence, in a circle, the centrifugal, and therefore centripetal force varies as the square of the velocity directly and the radius inversely.

Cor. 2. The centrifugal force in the curve PW is the same as in the circle nQ, the angular motions being the same in each. Now $Qn \propto \frac{\text{area } SPQ}{SP}$; hence, the centrifugal force $\frac{Qn^2}{SP}$ varies as $\frac{(\text{area } SPQ)^2}{SP^3}$.

Cor. 3. As the centripetal force of a body is measured by it's weight, when this is equal to the centrifugal force, the latter is represented by the weight of the body.

Cor. 4. As $nm = \frac{Qn^2}{2Ps}$; it appears that the centrifugal force varies as the square of the perpendicular velocity directly and the radius inversely; and when the angular velocity is given, the centrifugal force varies as the radius.

Prop. CXXXIX.

If the force of gravity upon the earth's surface be represented by $32\frac{1}{6}$ *feet, and* r *in feet represent the radius of any circle, about the centre of which a body revolves with the velocity* v, *and* F *represent the centripetal, and consequently the centrifugal force; then* $F = \frac{v^2}{r}$.

(202.) For let $V =$ the velocity of a body revolving in a circle at the earth's surface, about it's centre, $R =$ the radius of the earth; then $\frac{V^2}{2R} =$ the sagitta of the arc described in $1'' = 16\frac{1}{12}$ feet; and as the forces of bodies revolving in different circles vary as the squares of the velocities directly and the radii inversely (Cor. 1. last Prop.), $32\frac{1}{6} : F :: \frac{V^2}{R} : \frac{v^2}{r}$; but $32\frac{1}{6} = \frac{V^2}{R}$; hence, $F = \frac{v^2}{r}$.

If bodies of different magnitudes revolve about an axis, then in estimating the centrifugal force, the quantities of matter in the bodies must also be considered.

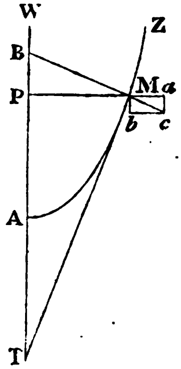
Cor. 1. If $r =$ radius of curvature of any curve; then the force being the same in the curve and the circle, the same is true for the curve, r being the radius of curvature.

Cor. 2. Let $p =$ the periodic time in a circle whose rad. $= r$, then v : circum. :: $1''$: p, and $v = \frac{\text{circum.}}{p} \propto \frac{r}{p}$; hence, $F \left(\frac{v^2}{r}\right) \propto \frac{r}{p^2}$. When p is given, $F \propto r$.

PROP. CXL.

Let AZ *be a slender rod revolving about an axis* AW *perpendicular to the horizon, and a ring moving freely on the rod be put any where upon it; to find the nature of the curve* AZ, *so that the ring may always remain at rest.*

Draw *PM* perpendicular to *AW*, and *BM* to the curve at *M*; produce *PM* to *a*, and let *Ma* represent the centrifugal force at *M*, and *Mb* perpendicular to it, the force of gravity; then the compound force *Mc* must be perpendicular to the curve, or lie in the direction *BM*, in order that the body may be urged neither up nor down the curve. Let $2m = 32\frac{1}{6}$ feet, $v =$ the velocity at the distance 1 foot from the axis, $x = AP$, $y = PM$, $d = PB$; then $vy =$ velocity of *M*; and (Prop. 139.) if $Mb = 2m$, $\frac{v^2y^2}{y} = v^2y = Ma$; and by sim. tri. $v^2y : 2m :: y : d = \frac{2m}{v^2}$ a constant quantity; hence, (Art. 24. Ex.) *AMZ* is a parabola whose latus rectum is $2d = \frac{4m}{v^2}$, and if *v* be given, the parabola is given.



Cor. 1. Hence, $y^2 = \frac{4m}{v^2} \times x$, or $y^2 v^2 = 4mx$; that is, the square of the velocity of the point $M = 4mx$; therefore (by *Mechanics*,) the velocity of that point is that which is acquired in falling through *PA*.

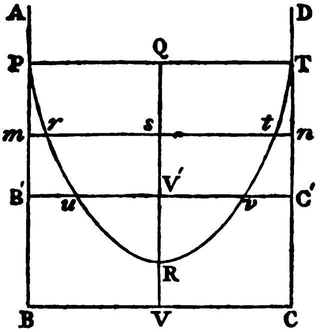
Cor. 2. Hence, if a vessel of water revolve uniformly about it's axis, the water will rise up in the curve of a parabola; for the water cannot rest, till the above-mentioned compounded force *MC* acts perpendicularly to the surface of the water.

Cor. 3. Let *AM* be an *ellipse*, one of whose axes (a) lies in the direction *AW*, and the other (b) perpendicular to it, and draw the tangent *MT*. Then $PT = \frac{2ax - x^2}{a - x}$; and to find at what point *M* the ring will remain at rest, we have (as in the Prop.) $v^2 y : 2m :: y : BP :: \frac{2ax - x^2}{a - x} : y$, and $v^2 y^2 = 2m \times \frac{2ax - x^2}{a - x} = 2m \times PT = 4m \times \frac{1}{2} PT$. Hence, the velocity of the point *M* is that which a body would acquire in falling through $\frac{1}{2} PT$; and $\frac{1}{2} PT$ being greater than *PA*, the velocity at *M* is *greater* than that in a parabola at *M*. As b is not here concerned, the velocity at *M* is the same, whatever be the axis perpendicular to *AW*. If *AMZ* be an *hyperbola*, the velocity at *M* is *less* than that in a parabola at *M*.

Prop. CXLI.

Let mn *be the surface of water when at rest in a cylindrical vessel* ABCD; QV *it's axis perpendicular to the horizon, and about which the vessel revolves with a given velocity; to find the position of the surface of the fluid.*

By Cor. 2. last Prop. the surface *PRT* of the fluid will put on the form of a parabola; and as the latus rectum (l) is given from the given velocity, and PQ is



given, we know QR. Put $Rs = x$, $PQ = r$, then $lx = rs^2$, $QR = a$, $Vs = b$, $p = 3{,}14159$, &c. then (Prop. 22. Ex. 1.) $\frac{1}{2}plx^2 =$ solid Rrt, $\frac{1}{2}pr^2a =$ solid PTR; hence, $\frac{1}{2}p \times (r^2a - lx^2) =$ solid $rPTt$; also, $pr^2 \times (a - x) =$ solid $PmnT$; therefore $pr^2 \times (a - x) - \frac{1}{2}p \times (r^2a - lx^2) =$ content of the elevated water above mn, $= \frac{1}{2}plx^2$ the water displaced; hence, $x = \frac{1}{2}a$, therefore $Qs = Rs$; that is, *the vertex* R *sinks as much below the original surface of the water, as the water at the highest point stands above it.*

Hence, (Cor. 1. last Prop.) the velocity of any point r is that which a body would acquire in falling down sR or Qs.

When $x = b$, or when $l = \frac{r^2}{2b}$, R comes to V. Now if the velocity be further increased, R will fall below the bottom of the vessel; let therefore $B'V'C'$ represent the bottom in such a case, b being now $V's$. Here, the water displaced $=$ solid $rtvu = \frac{1}{2}plx^2 - \frac{1}{2}pl \times (x - b)^2$; hence, $pr^2 \times (a - x) - \frac{1}{2}p \times (r^2a - lx^2) = \frac{1}{2}plx^2 - \frac{1}{2}pl \times (x - b)^2$, from which we get x, and thence the parabola is determined.

If the height of the vessel be given, and a given quantity (q) of water be thrown over the top, then q

must be added to the above-mentioned expression for the elevated water, and that made equal to the water displaced.

If n'' = the time of a revolution, then $v : 2p :: 1'' : n''$, and $v = \frac{2p}{n}$; hence, $l = \frac{4m}{v^2} = \frac{4mn^2}{4p^2} = \frac{mn^2}{p^2}$; therefore $\frac{mn}{p^2} \times x = y^2$ the parabola, determined from the time of revolution.

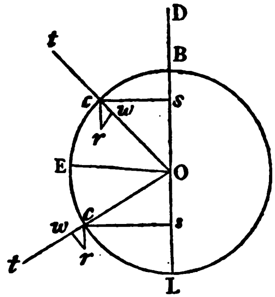
If the vessel be of any other form, we proceed in the same manner.

If the form of the vessel be that of a paraboloid whose latus rectum $= d$; then $a = \frac{db}{\sqrt{l^2 - dl}}$, and $x = \frac{b}{l} \times (l - \sqrt{l^2 - dl})$. Let $d = l$, then a is infinite and $x = b$; that is, if the height of the vessel were infinite, the fluid would sink to V, and rise up and just cover the whole inside. This would also appear without investigation, as (because $l = d$) the fluid puts on the same form as that of the vessel. But it is here supposed, that the force of gravity ($2m$) continues as at the earth's surface.

PROP. CXLII.

Let a stone in a sling be whirled round the centre O *in a vertical circle* BcL, B *the highest point, with a velocity at* B *just sufficient to keep it in the circle at that point; to find the force with which the string is stretched at any point* c.

Produce Oc to t, and let ct represent the centrifugal force; draw cs perpendicular to BL, and cr to cs, cr representing the force of gravity, and draw rw perpendicular to horiz.; produce OB to D, making $DB = \frac{1}{2} BO$.



Put $BO = r$, $DB = \frac{1}{2}r$, $v =$ velocity at c, $s = Bs$. Now, Cor. 2. Prop. 149. a body must fall from D to B to acquire the velocity in the circle at B. Then (by *Mech.*) $v^2 = 4m \times Ds = 4m \times (\frac{1}{2}r + s)$; and (Prop. 139.) if $cr = 2m =$ gravity, $ct = \frac{v^2}{r} = \frac{4m \times (\frac{1}{2}r + s)}{r}$; and (sim. tri.) $cr\,(2m) : cw :: r : r - s$, $\therefore cw = \frac{2mr - 2ms}{r}$; hence, c is drawn by a force $= \frac{4m \times (\frac{1}{2}r + s)}{r} - \frac{2mr - 2ms}{r} = \frac{6ms}{r}$; therefore gravity : force with which the string is stretched $:: 2m : \frac{6ms}{r} :: r : 3s$; and this is true when c falls below O, when the force $= ct + cw$.

Hence, at the lowest point L, the ratio is 1 : 6.

Let OE be parallel to the horizon, and the stone begin to descend from E, and put $Os = s$; then $v^2 = 4ms$, and $ct = \frac{4ms}{r}$; also $cr\ (2m) : cw :: r : s$, $\therefore cw = \frac{2ms}{r}$; hence, at any point c, the string is stretched by gravity just one half of what it is stretched by the centrifugal force.

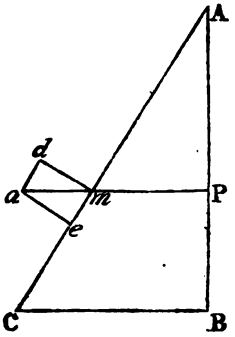
At B, $Cr = Cw$, and $Ct = 2m = Cr$; that is, the centrifugal force = the centripetal, or equal to the weight of the body. A body therefore revolving in a circle with a velocity equal to that acquired by falling through half the radius, acquires a centrifugal force equal to it's weight.

In like manner it appears, that when a pendulum vibrates between two semi-cycloids, beginning from the highest point, the force with which the string is stretched by gravity = the centrifugal force.

Prop. CXLIII.

Let a ring be put upon a slender rod AC, *and let the rod revolve about* AB *which is perpendicular to the horizon; it is required to find how long the ring will be in descending from* A *to* C, *the velocity of the rod, it's length, and the angle* CAB *being given.*

(203.) Draw CB perpendicular to AB; put $AB = a$, $BC = b$, $AC = c$, d = the velocity of the point C,



$x = Am$, v = the velocity of the ring at m, $m = 32\frac{1}{6}$ feet of the force of gravity, and t = the time of the ring's descent. Draw mP perpendicular to AB, and produce it to a, and let ma represent the centrifugal force of the point m; resolve ma into two forces, one *md* perpendicular to *AC*, and the other *me* in the direction *AC*. By similar triangles, $c : b :: x : \frac{bx}{c} = Pm$, and $b : \frac{bx}{c} :: d : \frac{dx}{c} =$ the velocity of the point *m*; hence (Art. 202.), the centrifugal force $ma = \frac{d^2x^2}{c^2} \times \frac{c}{bx} = \frac{d^2x}{bc}$; and by similar triangles, $c : b :: \frac{d^2x}{bc} : me = \frac{d^2x}{c^2}$; also, $c : a :: m$ (the force of gravity) $: \frac{ma}{c} =$ the accelerative force of the ring from the action of gravity; hence, (Art. 82. Cor.) $\frac{d^2x\dot{x}}{c^2} + \frac{ma\dot{x}}{c} = v\dot{v}$; and $v = \sqrt{\frac{d^2x^2}{c^2} + \frac{2max}{c}} = \left(\text{if } \frac{mac}{d^2} = n\right) \frac{d}{c}\sqrt{x^2 + 2nx}$. Hence (Art. 82.), $\dot{t} = \frac{c}{d} \times \frac{\dot{x}}{\sqrt{x^2 + 2nx}}$, and (Art. 45. Ex. 5.) $t = \frac{c}{d} \times$ h. l. $(n + x + \sqrt{x^2 + 2nx}) + C$; but when $x = 0$, $t = 0$, and we have $0 = \frac{c}{d} \times$ h. l. $n + C$; hence, the correct fluent $t = \frac{c}{d} \times$ h. l. $\frac{n + x + \sqrt{x^2 + 2nx}}{n} =$ (when $x = c$) $\frac{c}{d} \times$ h. l. $\frac{n + c + \sqrt{c^2 + 2nc}}{n}$ the whole time of descent.

Cor. 1. The accelerative force $\frac{d^2x}{c^2}$ of the ring in the direction of the rod, arising from the centrifugal force, is always the same whatever be the inclination of the rod, the length of the rod, and the velocity of it's lowest point being given.

COR. 2. By similar triangles, $c : a :: \frac{dx^2}{bc} : md = \frac{d^2ax}{c^2b}$; and by *Mechanics*, $c : b :: m : \frac{bm}{c}$ = the pressure of the ring on the rod; hence, when $\frac{d^2ax}{c^2b} = \frac{bm}{c}$, the pressure of the ring on the rod = 0, which therefore happens when $x = \frac{b^2cm}{d^2a}$.

COR. 3. If *AC* become horizontal, then $a = 0$, and $v\dot{v} = \frac{d^2x\dot{x}}{c^2}$. Now as in this case the ring will not begin to move from *A*, we must at first put it at some distance r from *A*. Hence, $v^2 = \frac{d^2x^2}{c^2} + C$, and when $v = 0$, $x = r$; therefore the equation becomes $0 = \frac{d^2r^2}{c^2} + C = 0$, and $C = -\frac{d^2r^2}{c^2}$; hence, $v = \frac{d}{c} \times \sqrt{x^2 - r^2}$. Also, $\dot{t} = \frac{c}{d} \times \frac{\dot{x}}{\sqrt{x^2 - r^2}}$, whose fluent (Art. 45. Ex. 4.) is $t = \frac{c}{d} \times \text{h. l. } (x + \sqrt{x^2 - r^2}) + C$; but when $t = 0$, $x = r$, and the equation becomes $0 = \frac{c}{d} \times \text{h. l. } r + C$; therefore $C = -\frac{c}{d} \times \text{h. l. } r$; hence, $t = \frac{c}{d} \times \text{h. l.} \frac{x + \sqrt{x^2 - r^2}}{r}$.

COR. 4. If *A* be the lower point of the rod, and *C* the higher; then the force $\frac{d^2x}{c^2}$ acts upwards, and the accelerating force of the ring $= \frac{d^2x}{c^2} \sim \frac{ma}{c}$. Let the ring at first be at any distance from *A*; then if $\frac{ma}{c}$ be greater than $\frac{d^2x}{c^2}$, the ring *descends* by the force $\frac{ma}{c} - \frac{d^2x}{c^2}$; but if $\frac{d^2x}{c^2}$ be greater than $\frac{ma}{c}$, the ring *ascends* by the force $\frac{d^2x}{c^2} - \frac{ma}{c}$; and the velocity and time may be found in each case as before.

Cor. 5. Taking the position of the rod as in the last Corol., and the case when $\frac{d^2x}{c^2}$ is greater than $\frac{ma}{c}$, let the ring at the distance r from *A* be projected downwards on the rod with the velocity e; then $v\dot{v} = \frac{d^2x\dot{x}}{c^2} - \frac{ma\dot{x}}{c}$, and $\frac{v^2}{2} = \frac{d^2}{c^2} \times \frac{x^2}{2} - \frac{max}{c} + C$; but when $v = e$, $x = r$, and the equation becomes, $\frac{e^2}{2} = \frac{d^2}{c^2} \times \frac{r^2}{2} - \frac{mar}{c} + C$, therefore $C = \frac{e^2}{2} - \frac{d^2}{c^2} \times \frac{r^2}{2} + \frac{mar}{c}$; hence, $v^2 = e^2 + \frac{d^2}{c^2} \times (x^2 - r^2) + \frac{2ma}{c} \times (r - x)$. Make $v=0$, and we get $x = \frac{mca}{d^2} + \sqrt{\frac{m^2c^2a^2}{d^4} + r^2 - \frac{c^2e^2}{d^2} - \frac{2macr}{d^2}}$, the distance from *A*, to which the ring descends when it has lost all it's velocity. If the value of x be impossible, the ring will come to *A* without losing all it's velocity. If the quantity under the radical sign $= 0$, $x = \frac{mca}{d^2}$;

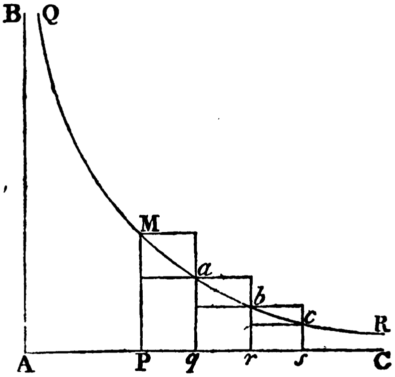
which is the value of x when the force $\frac{d^2x}{c^2} - \frac{ma}{c} = 0$; in this case therefore the ring will remain at rest when it has lost all it's velocity. If the quantity under the radical sign be positive, then when $v = 0$, the force $\frac{d^2x}{c^2} - \frac{ma}{c}$ acting upwards, the ring will return, and continue to ascend. Put $n = \frac{mca}{d^2}$, $p = \frac{c^2e^2}{d^2} + 2rn - r^2$; and we have $\dot{t} = \frac{c}{d} \times \frac{-\dot{x}}{\sqrt{x^2 - 2nx + p}}$; let $x - n = y$, and $x^2 - 2nx = y^2 - n^2 + p = y^2 + q^2$ (putting $-n^2 + p = q^2$); also, $\dot{x} = \dot{y}$; hence, $\dot{t} = \frac{c}{d} \times \frac{-\dot{y}}{\sqrt{y^2+q^2}}$, and $t = \frac{c}{d} \times -$ h. l. $(y + \sqrt{y^2 + q^2}) + C$; but when $t = 0$, $x = r$, $\therefore y = r - n$; and the fluent becomes $0 = \frac{c}{d} \times -$ h. l. $\left(r - n + \sqrt{(r-n)^2 + q^2}\right) + C$, and $C = \frac{c}{d} \times$ h. l. $\left(r - n + \sqrt{(r-n)^2 + q^2}\right)$; hence, $t = \frac{c}{d} \times$ h. l. $\frac{r - n + \sqrt{(r-n)^2 + q^2}}{y + \sqrt{y^2 + q^2}} = \frac{c}{d} \times$ h. l. $\frac{r - n + \sqrt{(r-n)^2 + q^2}}{x - n + \sqrt{(x-n)^2 + q^2}}$, the time of descent.

On the same principle we may find the motion of a ring on a *curve* line revolving in like manner.

Prop. CXLIV.

To show when the series $\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} +$ &c. ad infinitum *is finite, and when infinite.*

(204.) Let QR be an hyperbolic curve between the asymptotes AB, AC, which are perpendicular to each other; take AP = ordinate $PM = 1$, and let Pq, qr, rs, &c.



be each $= 1$, and draw the ordinates qa, rb, sc, &c. and complete the circumscribing parallelograms qM, ra, sb, &c. and the inscribed Pa, qb, rc, &c. and let the ordinate be equal to the inverse n^{th} power of the abscissa; then will $PM = \frac{1}{1^n}$, $qa = \frac{1}{2^n}$, $rb = \frac{1}{3^n}$, $sc = \frac{1}{4^n}$, &c. and as the basis of these parallelograms are each $= 1$, the area of the parallelogram $qM = \frac{1}{1^n}$, of $ra = \frac{1}{2^n}$, of $sb = \frac{1}{3^n}$, &c. therefore the sum of all the circumscribed parallelograms $= \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} +$ &c. *ad infinitum;* but it is manifest that the sum of all the inscribed parallelograms is less than the sum of all the circumscribed parallelograms, by the first parallelogram qM, that parallelogram being the sum of all the parallelograms Ma, ab, bc, &c. each of which expresses the difference between it's respective inscribed and circumscribed parallelogram. But the whole curvilinear area *PMRC* (being between the sum of the inscribed and circumscribed parallelograms) is less than the sum of all the circumscribed parallelograms, by a quantity which is less than the parallelogram *qM*; these two therefore differing by a finite quantity, when one is finite the other is finite, and when one is infinite the other is infinite. But by Prop. 20. Ex. 3. when n is equal to or less than unity, the area of the curve is infinite, and when n is greater than unity, the area is finite. Hence, the sum of the given series is *infinite* when n is *equal* to or *less* than unity, and *finite* when n is *greater* than unity.

Prop. CXLV.

If $\frac{x^m + ax^{m-1} + \&c.}{x^n + px^{n-1} + \&c.}$ *be the general term of a series, and for* x *we write* 1, 2, 3, &c. in infinitum, *then if* n *be greater than* $m + 1$, *the sum of the series is finite.*

Take each term in the numerator separately, and first assume $\frac{x^m}{x^n} = \frac{1}{x^{n-m}}$; then as $n - m$ is greater than 1, by the last Prop. the sum of the series will be finite. Now if the sum be finite on this supposition, *a fortiori*, it must be finite when you take in all the terms of the denominator, as that must diminish the value of each term. This being the case when you take the first term of the numerator, it must necessarily be the case when you take each of the other terms, they being less than the first. And as the sum of all the sums must give the sum for the general term, that sum must be finite.

For the same reason (referring to the last Prop.) if n be not greater than $m + 1$, the sum must be infinite.

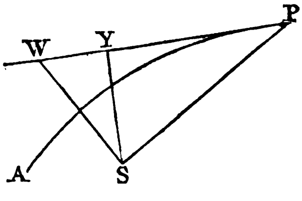
Cor. Assume $\dfrac{(x+a)\times(x+b)\times\text{\&c.}\ldots\text{to } m \text{ terms}}{(x+p)\times(x+q)\times\text{\&c.}\ldots\text{to } n \text{ terms}}$ the general term of a series, then the sum will be finite if n be greater than m by 2; for if these factors be multiplied together, we shall have the same dimensions of x in the numerator and denominator as before, and n is greater than $m+1$. If n be not greater than m by 2, the sum will be infinite.

If for x we write 1, 3, 5, &c. successively, or any set of numbers at equal intervals from each other, the sum of the series will, under the same circumstances, be finite or infinite, as is evident from the last Prop. by assuming the breadths Pq, qr, rs, &c. accordingly, so as to give the series.

Prop. CXLVI.

To determine the law of centripetal force tending to S, *so that a body may describe any given curve* AP.

(205.) Let SY be perpendicular to the tangent PY, and P the place of the body. Put $x = SP$, $u = SY$, $F =$ force in the direction PS, $f =$ that part of F which acts in the direction PY, $v =$ the velocity at P, and $z = AP$. Now (Art. 81. Cor.) $v\dot{v} = f\dot{z}$; but $F : f :: SP : PY ::$ (Art. 32.) $\dot{z} : \dot{x}$, therefore $f\dot{z} = F\dot{x}$; hence,



$v\dot{v} = F\dot{x}$, or rather $v\dot{v} = -F\dot{x}$*, because (Art. 16.) when

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* If the force of gravity be represented by $2m$, then $v\dot{v} = -2mF\dot{x}$, and $F = \dfrac{-v\dot{v}}{2m\dot{x}}$.

v increases x decreases; therefore $F = \frac{-v\dot{v}}{\dot{x}}$. But (*Newton's Prin. L.* 1. Pr. 1. Cor. 1.) $v \propto \frac{1}{u}$; therefore $v\dot{v} \propto \frac{-\dot{u}}{u^3}$; hence, $F \propto \frac{\dot{u}}{u^3\dot{x}}$.

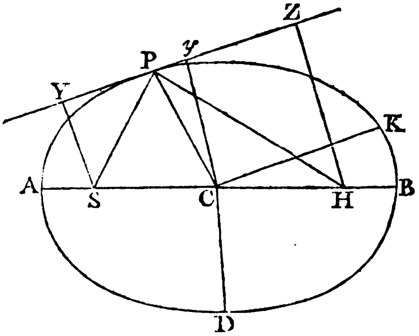
(206.) Cor. Hence, whatever be the angle SPY, if v remain the same, then if $\dot{v}$ be given, $\dot{x}$ will be given; and if we suppose the angle SPY to vanish, it follows, that if the velocity (v) of a body in the curve at P be equal to the velocity of a body in the right line SP at P, they will be equal at all other equal distances from S.

Ex. 1. Let AP be the *logarithmic* spiral, S it's centre. Then $x : u :: a : b$ some constant ratio, $\therefore \dot{x} = \frac{a}{b}\dot{u}$; hence, $F \propto \frac{b}{a} \times \frac{1}{u^3} \propto \frac{1}{x^3}$.

Ex. 2. Let AP be the *hyperbolic* spiral. Draw SW perpendicular to SP, meeting the tangent at W; then by the property of the curve, $SW = a$ a constant quantity; and $WP = \sqrt{a^2+x^2}$; hence, by similar triangles, $\sqrt{a^2+x^2} : x :: a : u = \frac{ax}{\sqrt{a^2+x^2}}$, and $\frac{1}{u^2} = \frac{1}{x^2} + \frac{1}{a^2}$, therefore $\frac{\dot{u}}{u^3} = \frac{\dot{x}}{x^3}$; hence, $F \propto \frac{\dot{u}}{u^3\dot{x}} \propto \frac{1}{x^3}$.

Ex. 3. Let APB be an *ellipse* whose focus is S; let H be the other focus, C the centre, CD the semi-axis minor, and HZ perpendicular to PY. Put $a = AC$, $b = CD$; then $2a - x = PH$, and by sim. tri. $x : u :: 2a - x : HZ = \frac{(2a-x) \times u}{x}$, and (*Con. Sect.* p. 6.)

$\frac{(2a-x)\times u^2}{x} = b^2$; hence, $\frac{1}{u^2}=\frac{2a}{b^2x}-\frac{1}{b^2}$, and $\frac{\dot{u}}{u^3} =$



$\frac{a\dot{x}}{b^2x^2}$; therefore $F\propto\frac{\dot{u}}{u^3\dot{x}}\propto\frac{a}{b^2x^2}\propto\frac{1}{x^2}$.

For an *hyperbola*, $2a+x=PH$, and the same conclusion follows.

For a *parabola*, $x\propto u^2$ (*Con. Sect.* p. 8. Cor. 2.), therefore $\frac{1}{u^2}\propto\frac{1}{x}$, and $\frac{\dot{u}}{u^3}\propto\frac{\dot{x}}{x^2}$; hence, $F\propto\frac{\dot{u}}{u^3\dot{x}}\propto\frac{1}{x^2}$.

Hence, a force tending to the focus of any of the conic sections, varies in the inverse duplicate ratio of the distance. And *vice versa*, if the force $\propto\frac{1}{x^2}$ the body will describe an ellipse about the focus; for if there be given the force at *P*, velocity and direction, we can thence construct an ellipse (see my *Conic Sections*); hence, an ellipse may be constructed in which a body *may* move, and therefore a body *must* move in it; for with the same data, a body cannot describe two different curves.

Ex. 4. Let the force tend to the centre *C* of the ellipse. Let *CK* be the semi-conjugate to *CP*, and *Cy* perpendicular to *Py*; $CP=x$, $Cy=u$; then (*Con. Sect.*

p. 13.) $a^2 + b^2 = x^2 + CK^2$, and $CK = \sqrt{a^2 + b^2 - x^2}$; also, (*Con. Sect.* p. 11.) $ab = u \times \sqrt{a^2 + b^2 - x^2}$, and $u^2 = \frac{a^2 b^2}{a^2 + b^2 - x^2}$, therefore $\frac{1}{u^2} = \frac{1}{b^2} + \frac{1}{a^2} - \frac{x^2}{a^2 b^2}$, and $\frac{\dot{u}}{u^3} = \frac{x\dot{x}}{a^2 b^2}$; hence, $F \propto \frac{\dot{u}}{u^3 \dot{x}} \propto \frac{x}{a^2 b^2} \propto x$. Hence, *vice versa*, as before.

For an *hyperbola*, $F \propto -x$, which shows the force to be repulsive.

Ex. 5. Let it be the *spiral* in Article 32. Here $SY^2 = \frac{m^2 x^{2m+2}}{t^{2m} + m^2 x^{2m}}$, and $\frac{1}{SY^2}$ or $\frac{1}{u^2} = \frac{t^{2m}}{m^2 x^{2m+2}} + \frac{1}{x^2}$, therefore $\frac{\dot{u}}{u^3} = \frac{(m+1) \times t^{2m} \dot{x}}{m^2 x^{2m+3}} + \frac{\dot{x}}{x^3}$; hence, $F \propto \frac{\dot{u}}{u^3 \dot{x}} \propto \frac{(m+1) \times t^{2m}}{m^2 x^{2m+3}} + \frac{1}{x^3}$.

If $m = 1$, it is the spiral of *Archimedes*, and $F \propto \frac{2t^2}{x^5} + \frac{1}{x^3}$.

If $m = -1$, it is the *reciprocal* spiral, and $F \propto \frac{1}{x^3}$.

If $m = -2$, it is the *Lituus*, and $F \propto -\frac{x}{2t^4} + \frac{1}{x^3}$. When the negative part is *greater* than the positive, the force is repulsive, and the curve is convex to the centre; when it is *less*, the force is attractive, and the curve is concave to the centre; but at the point of contrary flexure $F = 0$, or $\frac{-x}{4t^4} + \frac{1}{x^3} = 0$, and $x = t\sqrt{2}$, as found in Art. 80. And like circumstances must take place in all cases where $m + 1$ is negative.

Prop. CXLVII.

The velocity of a body revolving in any curve about a centre of force : velocity of a body revolving in a circle at the same distance, in the sub-duplicate ratio of the chord of curvature :: twice the distance, or in the sub-duplicate ratio of $\frac{\dot{\mathrm{x}}}{\mathrm{x}} : \frac{\dot{\mathrm{u}}}{\mathrm{u}}$.

(207.) For (Art. 97.) let *sr* be a sagitta of a circle of curvature to any curve, parallel to the chord *CV* which passes through the centre of force; then by sim. tri. *sr* : *Cr* :: *Cr* : *CV*, but *Cr* : the arc *Cr* ultimately in a ratio of equality; therefore ultimately, *sr* : arc *Cr* :: arc *Cr* : *CV*; hence, arc $Cr = \sqrt{sr \times CV}$; but *sr*, *dato tempore*, is as the force, and *Cr* is as the velocity; therefore the velocity $\propto \sqrt{\text{force} \times \text{chord curvature}}$; but at the same distance, the force is the same in the circle and in the curve, and the chord of curvature of the circle is it's diameter, or twice the distance; therefore the velocity in the curve : velocity in the circle :: $\sqrt{\text{ch. curv. of the curve}} : \sqrt{\text{twice dist.}}$. But the chord of curvature (Art. 101.) is $\frac{2u\dot{x}}{\dot{u}}$; hence, the velocity in the curve : velocity in the circle :: $\sqrt{\frac{2u\dot{x}}{\dot{u}}} : \sqrt{2x} :: \sqrt{\frac{\dot{x}}{x}} : \sqrt{\frac{\dot{u}}{u}}$.

Ex. 1. Let the curve be the *logarithmic* spiral. Here, the velocities are equal, because the chord of curvature = twice the distance; or, as $u \propto x$, therefore $\frac{\dot{x}}{x} = \frac{\dot{u}}{u}$.

Ex. 2. Let the curve be an *ellipse* with the force tending to the *focus*. Here, (Art. 206. Ex. 3.) $\frac{\dot{u}}{u^3} = \frac{a\dot{x}}{b^2x^2}$; hence, $\frac{\dot{x}}{x} : \frac{\dot{u}}{u} :: \frac{1}{u^2} : \frac{a}{b^2x} ::$ $\left(\text{as } \frac{1}{u^2} = \frac{2a - x}{b^2x}\right)$ $2a - x : a$; therefore the velocity in the ellipse : velocity in the circle :: $\sqrt{2a - x} : \sqrt{a} :: \sqrt{PH} : \sqrt{AC}$. Hence, the velocities are equal at the extremity of the minor axis.

Ex. 3. Let the force tend to the *centre* of the ellipse. Here, (Art. 206. Ex. 4.) $\frac{\dot{u}}{u^3} = \frac{x\dot{x}}{a^2b^2}$; hence, $\frac{\dot{x}}{x} : \frac{\dot{u}}{u} :: \frac{1}{u^2} : \frac{x^2}{a^2b^2} ::$ (as $a^2b^2 = u^2 \times CK^2$) $CK^2 : x^2$; therefore the velocity in the ellipse : velocity in the circle :: $CK : x$, or CP. Hence, the velocities are equal when $CK = CP$.

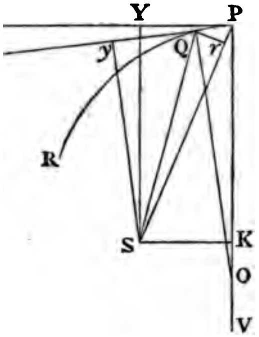
Ex. 4. Let the curve be the *hyperbolic* spiral. Here, (Art. 206. Ex. 2.) $\frac{\dot{x}}{x^3} = \frac{\dot{u}}{u^3}$; hence, $\frac{\dot{x}}{x} : \frac{\dot{u}}{u} :: \frac{1}{u^2} : \frac{1}{x^2}$ $:: x^2 : u^2$; therefore the velocity in the curve : velocity in the circle :: $x : u$.

Prop. CXLVIII.

Let PR *be any curve,* S *any point in it's plane, join* SP, *and draw the perpendicular* SY *on the tangent* PY; *then if* P *move along the curve, to find the angular velocity of* SP *to that of* SY.

Take PQ indefinitely small, and draw the tangent Qy, and Sy perpendicular to it; to the curve at P, Q, draw PO, QO, perpendicular, and O is the centre of curvature; produce (if necessary) PO, and draw SV perpendicular to SP meeting PO in V, and SK to PV.

Now as PO, QO are respectively parallel to SY, Sy,



the angle $POQ = YSy$; we have therefore to compare the angle PSQ with POQ. Now $PSQ : POQ$ :: $\frac{Qr}{SP} : \frac{PQ}{PO}$ :: (because $Qr : PQ :: SY : SP$) $\frac{SY}{SP} : \frac{SP}{PO}$ :: $SY \times PO : SP^2$ :: $PK \times PO : PK \times PV$ :: $PO : PV$ :: (if $SP = y$, $PK = v$) $\frac{y\dot{y}}{\dot{v}}$ (Art. 101.) : $\frac{\dot{y}^2}{v}$ :: $\frac{\dot{v}}{v} : \frac{\dot{y}}{y}$ :: (Prop. 147.) the square of the velocity of a body revolving in a circle at P about S : square of the velocity of a body in the curve at P.

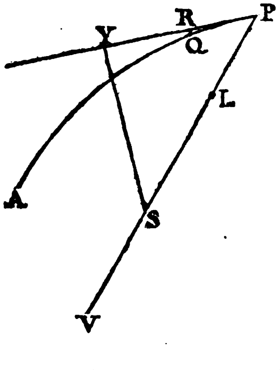
Cor. 1. Hence, when these velocities are equal, the angular velocities of SP, SY are equal. They are therefore always equal in the logarithmic spiral. In an ellipse about the focus, they are equal at the extremity of the minor axis; about the centre, they are equal at the points where the two diameters are equal; in the parabola, $PV = 2PO$; therefore the angular velocity of SY is double that of SP.

Cor. 2. Let the body set out from an apside, and the angles described by SP, SY, be x and y; then the angle $PSY = y - x$, and when this is a max. $\dot{y} - \dot{x} = 0$, and $\dot{x} = \dot{y}$, or when $POQ = YSy$, or when the velocity in the curve is equal to the velocity in a circle at the same distance; or when $PV = 2PS$. Hence, in an ellipse about the focus, the angle PSY is a max. at the extremity of the minor axis; about the centre, when the distance is equal to it's semi-conjugate diameter. In the parabloa and hyperbola, the angle never becomes a maximum.

Prop. CXLIX.

If a body revolve in any curve, the velocity (V) *at any point, is equal to the velocity which a body would acquire in falling down one-fourth of the chord of the circle of curvature passing through the centre of force, supposing the force to remain constant.*

(208.) By Prop. 45. in the limiting state of the arc PQ, $RQ : QP :: QP : PV = \frac{QP^2}{RQ}$. Now whilst PQ is described by the velocity V, the body is drawn by the force through RQ, and acquires a velocity (v) which,



in the same time, would, if continued uniform, make it pass over $2RQ$; and let PL be the space fallen through with the constant force at P, to acquire the velocity V. Then

$$\begin{array}{l} V^2 : v^2 :: PQ^2 : 4RQ^2 \\ v^2 : V^2 :: RQ : PL, \text{ by } \textit{Mechanics}, \\ \hline \therefore 1 : 1 :: PQ^2 : 4RQ \times PL; \end{array}$$

hence, $PL = \frac{PQ^2}{4RQ} = \frac{1}{4}PV.$

(209.) Cor. 1. Hence (by *Mechanics*), $V = \sqrt{2F \times \frac{1}{4}PV} = \sqrt{F \times \frac{1}{2}PV}$; therefore $V \propto \sqrt{F \times PV}$.

Cor. 2. Hence, if the curve be a circle, and the centre of force in the centre, a body must fall down half the radius. If $r =$ the radius of the earth, $m = 16\frac{1}{12}$ feet, $V = \sqrt{2mr}$ the velocity necessary to carry a body round the earth.

Prop. CL.

If a body revolve in a circle about the centre, to find it's velocity.

(210.) Let the force of gravity on the earth's surface be denoted by unity, the radius of the earth by unity, and the velocity of a body revolving about the earth at it's surface by unity; and in proportion to these, let $x =$ the radius of any circle, $v =$ the velocity of a body revolving in that circle, and the force $= x^n$; then as a body must fall down $\frac{1}{2}$ of the radius to acquire the velocity in the circle, the force remaining constant, and by *Mechanics*, the velocity varies as the square root of the force and space conjointly, we have $1 : \sqrt{1 \times \frac{1}{2}} :: v : \sqrt{x^n \times \frac{1}{2}x}$; hence, $v = x^{\frac{n+1}{2}}$.

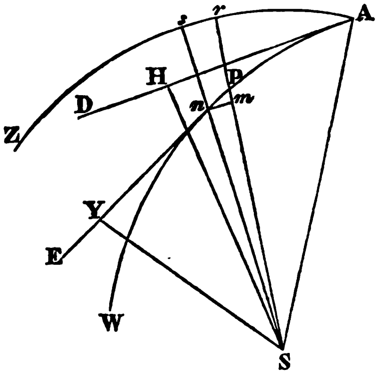
(211.) Cor. As the periodic time (P) varies as the circumference of the circle directly and velocity (v) inversely, and therefore as the radius (x) directly and v inversely, we have $P \propto \frac{x}{x^{\frac{n+1}{2}}} \propto x^{\frac{1-n}{2}}$.

If $n=0$, $P \propto x^{\frac{1}{2}}$. If $n=1$, $P \propto x^{0}=1$, or P is constant. If $n=-2$, $P \propto x^{\frac{3}{2}}$.

Prop. CLI.

Given the law of force as any power of the distance, to find the curve which the body describes.

(212.) Let S be the centre of force, and let the body be projected in the direction AD, and describe the



curve APW; describe the circular arc AZ with the centre S; draw the tangent PE, on which let fall the perpendicular SY, and SH on AD; also draw Sn indefinitely near to SP, and nm perpendicular to SP, and produce SP, Sn, to r and s. Put $SA=a$, $SH=p$, $Ar=z$, $SP=x$, $b=$ the velocity at A, $v=$ the velocity at P, $Pm=\dot{x}$, $rs=\dot{z}$. Now the velocity being inversely as the perpendicular, $v : b :: p : SY=\frac{pb}{v}$; therefore $PY=\sqrt{x^2-\frac{p^2b^2}{v^2}}=\frac{\sqrt{x^2v^2-p^2b^2}}{v}$; and by sim. trian. $\frac{\sqrt{x^2v^2-p^2b^2}}{v} : \frac{pb}{v} :: \dot{x} : mn=\frac{pb\dot{x}}{\sqrt{x^2v^2-p^2b^2}}$;

hence, $x : a :: \frac{pb\dot{x}}{\sqrt{x^2v^2-p^2b^2}} : \dot{z} = \frac{pab\dot{x}}{x\sqrt{x^2v^2-p^2b^2}}$ expressing the fluxional equation of the curve in terms of the angle described and distance. But (Art. 82. Ex. 7.) $v^2 = b^2 + \frac{2}{n+1} \times (a^{n+1} - x^{n+1})$; or if b (the vel. of proj.) : vel. $\left(a^{\frac{n+1}{2}}\right)$ in a circle at the same distance (Art. 210.) :: $m : 1$, then $b^2 = m^2a^{m+1}$; hence, $v^2 = \left(m^2 + \frac{2}{n+1}\right) \times a^{n+1} - \frac{2}{n+1} \times x^{n+1}$; therefore $\dot{z} =$

$$\frac{pab\dot{x}}{x\sqrt{\left(m^2+\frac{2}{n+1}\right)\times a^{n+1}\times x^2 - \frac{2}{n+1}\times x^{n+3} - p^2m^2a^{m+1}}}$$

the fluent of which can only be found in particular cases.

(213.) At the apsides, $SP = SY$, or $x = \frac{pb}{v} =$

$$\frac{pb}{\sqrt{\left(m^2+\frac{2}{n+1}\right)\times a^{n+1} - \frac{2}{n+1}\times x^{n+1}}};$$ hence,

$x\sqrt{\left(m^2+\frac{2}{n+1}\right)\times a^{n+1} - \frac{2}{n+1}\times x^{n+1}} - pb = 0$ the equation to the apsides. Now to find the *number* of apsides, by squaring the first equation, we get $\left(m^2+\frac{2}{n+1}\right) \times a^{n+1} \times x^2 - \frac{2}{n+1} \times x^{n+3} - p^2b^2 = 0$, which equation (*Algebra*, Art. 361.) may have 4 possible roots when n is an even number, and 3 when n is an odd number; but this being the square of the original equation, some of the roots are introduced by that operation, and the equation to the apsides can never have more than 2 possible roots, so that no orbit under this general law of force, can have more than 2 apsides, that is, there are only two different distances of the apsides ; but there is no limit to the number of repetitions of these, without their falling upon the same points. If n be -3, or a greater negative number, the equation can have only 1 possible root, and the orbit but one apside.

Prop. CLII.

Given the law of force as before, to find at what point of the curve the motion towards the centre is a maximum.

By the last Prop. $mn = \frac{pb\dot{x}}{\sqrt{x^2v^2 - p^2b^2}}$; now when the time is given, the area (A) is given, and $mn = \frac{2A}{x}$; hence, $\dot{x} = \frac{2A}{pb} \times \sqrt{v^2 - p^2b^2x^{-2}} = \frac{2A}{pb} \times \sqrt{\left(m^2 + \frac{2}{n+1}\right) \times a^{n+1} - \frac{2}{n+1} \times x^{n+1} - p^2b^2x^{-2}} =$ maximum, or $\frac{2}{n+1} \times x^{n+1} + p^2b^2x^{-2} =$ minimum, and $2x^n\dot{x} - 2p^2b^2x^{-3}\dot{x} = 0$; hence, $x = \overline{pb}|^{\frac{2}{n+3}}$ the distance required.

Because $v : b :: \frac{1}{SY} : \frac{1}{p}$, we have $pb = SY \times v$; hence, $x = \overline{SY \times v}|^{\frac{2}{n+3}}$ and $v = \frac{x^{\frac{n+3}{2}}}{SY}$. But if $c =$ chord of curvature passing through S, as (Prop. 150.) the velocity in a circle at $P = x^{\frac{n+1}{2}}$, we have (Prop. 147.) $v : x^{\frac{n+1}{2}} :: \sqrt{c} : \sqrt{2x}$ and $x^{\frac{n+1}{2}} \times \sqrt{\frac{c}{2x}} = v = \frac{x^{\frac{n+3}{2}}}{SY}$; hence, $2x^3 = SY^2 \times c$; but (Prop. 138.) this gives the point where the centrifugal force = centrifugal. Hence, the motion towards the centre is a maximum at the point where these forces are equal.

Referring to Fig. Art. 206. Ex 3. let the force tend to the *focus* S; then $c = \frac{2\,CK^2}{AC}$, and $SY^2 = DC^2 \times \frac{SP}{HP}$; hence, $2\,SP^3 = DC^2 \times \frac{SP}{HP} \times \frac{2\,CK^2}{AC}$; and as $SP \times HP = CD^2$, we have $SP \times AC = CD^2$; therefore P is at the extremity of the ordinate to the major axis, passing through S.

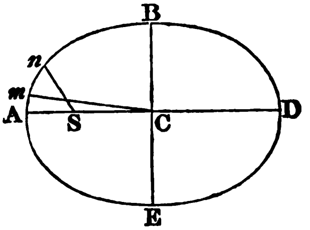
If the force be in the *centre* C, $c = \frac{2\,CK^2}{CP}$; and $2\,CP^3 = CY^2 \times \frac{2\,CK^2}{AC}$; hence, $CP^4 = CY^2 \times CK^2 = AC^2 \times BC^2$, and $AC : CP :: CP : BC$.

Prop. CLIII.

If two bodies revolve in an ellipse, one about the centre and the other about the focus, in the same periodic time; to compare the forces towards these centres.

Let $ABDE$ be an ellipse, S the focus, C the centre, AD the major, BE the minor axis. Now when the forces of central bodies vary in different ratios, we must get the relative effects at some one distance, in order to compare them at all other distances. Let $SA = s$, $CA = a$, and let the indefinitely small arcs Am, An, be described in the same time about C and S respectively;

and at the distance *d* let the effects of the two forces be



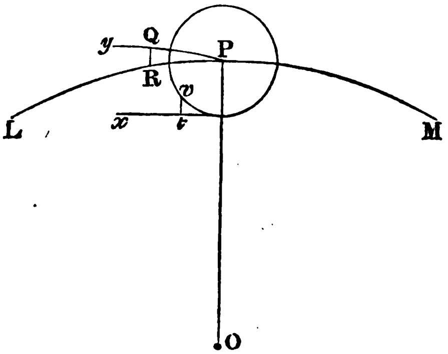
equal and represented by unity; then $\frac{1}{d^2} : \frac{1}{s^2} :: 1 : \frac{d^2}{s^2}$ the force at *A* about *S*, and $d : a :: 1 : \frac{a}{d}$ the force at *A* about *C*. Now (Prop. 147.) the velocities vary as the square root of the forces and chords of curvature; but the velocities are as *An*, *Am*, and the chord of curvature at *A* is the same for each force; hence, $An : Am :: \sqrt{\frac{d^2}{s^2}} : \sqrt{\frac{a}{d}}$; but as the periodic times are equal, equal areas are described in equal times; therefore $An \times AS = Am \times AC$, or $\sqrt{\frac{d^2}{s^2}} \times s = \sqrt{\frac{a}{d}} \times d$; hence, $d = a$, and the forces are equal at the extremity of the semi-axis major.

In B. I. Ch. 2. Prop. 11. of NEWTON'S PRINCIPIA, the second demonstration goes upon supposition that the periodic times about *S* and *C* are equal; this Prop. therefore shows how the forces must be here adjusted.

PROP. CLIV.

When a Satellite is in conjunction with the Sun, to find when it's orbit in fixed space is concave and when convex to the Sun.

Let *P* be the planet in it's orbit *LM*, *S* the satellite in conjunction with the sun at *O*; draw *Sx** a tangent



to the satellite's orbit at *S*, *Py*† a tangent to *LM*; *PR*, *Sv*, two indefinitely small arcs described in the same time, and draw *QR*, *vt* perpendicular to *Py*, *Sx*. Now if the curve described by *S* lie *below* *Sx*, the curve will be *concave* to *O*; if it lie *above* *Sx*, it will be *convex*. But as the planet passes through *PR*, it approaches *Sx* by *QR*; and carrying the orbit of the satellite with it, it carries the satellite with it in the same direction through the same space, or it tends to carry it so much on the side of *Sx* towards *O*; but the satellite by the motion in it's orbit, tends to carry itself through *tv* on the other side of *Sx*. Hence, the path of the satellite at *S* will be concave or convex, as *QR* is greater or less than *tv*. Let $P =$ periodic time of the planet, p that of the satellite, $D = PO$, $d = PS$; then (Prop. 139. Cor. 2.) $QR : tv :: \frac{D}{P^2} : \frac{d}{p^2} :: \frac{D}{d} : \frac{P^2}{p^2}$; hence, the orbit will be convex or concave toward the sun, as $\frac{D}{d}$ is less or greater than $\frac{P^2}{p^2}$.

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* *S* is where *xt* touches the circle.

† *Py* should be a straight line.

For the moon $\frac{D}{d} = 392$, $\frac{P^2}{p^2} = 178$; hence, the moon's orbit is concave to the sun.

For the first satellite of Jupiter $\frac{D}{d} = 1844$, $\frac{P^2}{p^2} = 5992704$; hence, it's orbit is convex to the sun. The same is true for the other satellites.

Prop. CLV.

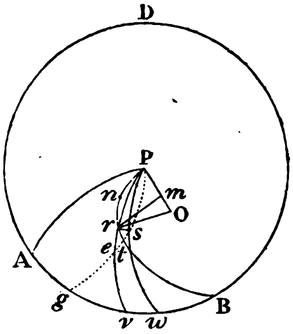
There is a field in the form of a parabola whose abscissa is x *and parameter* p; *now corresponding to the abscissa* e, *the land is worth* m *pounds an acre, and upon every other ordinate the value varies as the* n^th^ *power of the abscissa; what is the value of the whole field?*

Put x = any abscissa, y it's whole ordinate, $c = 160$ square rods = 1 acre; then $a^n : x^n :: m : \frac{mx^n}{a^n}$ the value of an acre at the ordinate y; hence, $c : y\dot{x} :: \frac{mx^n}{a^n} : \frac{m}{ca^n} \times yx^n\dot{x} = (\text{as } y = 2p^{\frac{1}{2}}x^{\frac{1}{2}})\, \frac{2mp^{\frac{1}{2}}}{ca^n} \times x^{\frac{2n+1}{2}}\dot{x}$, the fluxion of the value, whose fluent is $\frac{4mp^{\frac{1}{2}}}{ca^n \times (2n+3)} \times x^{\frac{2n+3}{2}}$, the whole value.

Prop. CLVI.

Let ABD *represent an hemisphere whose pole is* P, *and let a meridian* P r v *set out from* PA *and move uniformly about* P, *whilst a point* r *sets out from* P *and move uniformly along* P v *with a velocity which is to the velocity of the point* v *as* 1 : n; *and let* P n r B *be the curve described by the point* r; *to find the area* P A B r n P.

Draw the meridian *Pw* indefinitely near *Pv*, and *rs* parallel to *vw*; let *O* be the centre of the sphere, and perpendicular to *PO* draw *rm*; then the limit of the area *vrtw* is *vrsw* which is the fluxion of the zone corresponding to *Av*, *vr*. Let $r = rO = PO$, $p = 3{,}14159$. &c. $Pnr = z$, $rm = x$, $Om = y$, $a =$ area of the surface of the hemisphere; then $Av = nz$, $vw = n\dot{z}$. Now (Prop. 26. Ex. 1.) the surface corresponding to *Om* is as *Om*; hence, $r : y :: a : \frac{ay}{r}$ the area of the zone whose breadth is *rv*, *rv* corresponding to *Om* or *y*; and $2pr$ (the circumference *ABDA*) : $n\dot{z} :: \frac{ay}{r} : \frac{na}{2pr^2} \times y\dot{z} =$ (Art. 46. as $y\dot{z} = r\dot{x}$, and $a = 2pr^2$) $nr\dot{x}$ the area *vsrw*, whose fluent when $x = r$ is nr^2 the whole area *PABrnP*.



As $n : 1 :: AB : AP$, if $n = 4$, when *r* comes to *B*, *B* will have described the circumference and come to *A*, and the area $= 4r^2$ the square of the diameter.

These curious properties were discovered by Pappus. But the Problem may be extended by making the velocities of the points *r*, *v*, follow different laws.

For example, let rt $(\dot{z}) : vw :: y : x$, then $vw =$

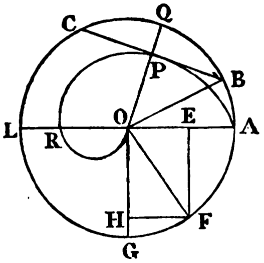
$\frac{x\dot{z}}{y}$; hence, $2pr : \frac{x\dot{z}}{y} :: \frac{ay}{r} : \frac{a}{2pr^2} \times x\dot{z} =$ (as $2pr^2 = a$, $x\dot{z} = -r\dot{y}$) $-r\dot{y}$, whose fluent is $-ry + C$ the area $PABrnP$; but when that area $= 0$, $y = r$, and $-r^2 + C = 0$, and $C = r^2$; hence, the whole area (y being then $= 0$) $= r^2$.

Let us suppose the point r to move just above the surface of the earth, from the pole to the equator with the velocity as in the Proposition, to find the curve on the surface, over which r passes. Whilst the point r moves from r to e, let the point of the earth under r move from r to s, then as the point was vertical to s, it is now vertical to e, and se will be the path on the surface over which the point moves; and the curve $Pseg$ so described will be the curve required; which is manifestly like to $PntB$, only lying the contrary way. Now $re : vw :: 1 : n$, and $vw : rs :: 1$ (rad.) : c (cos. lat.); therefore $1 : cn :: re : rs ::$ rad. : tan. rse which the curve makes with the meridian. This explains the cause of the *Trade Winds.* The cold air sets in from the north towards the equator, moving over the surface D of the earth in the manner here described; and the earth moving from A to B, or from west to east, the current of the air over the surface in the direction $Pseg$ makes a north-easterly wind on the north side of the equator; and for the like reason, a south-easterly on the south side. If the greatest velocity of wind towards the equator be 60 miles in an hour, $1 : n :: 1 : 17{,}4$ the tangent of $rse = 85°.\ 45'$ at the equator, the least angle the wind can there make with the equator.

Prop. CLVII.

If two equal bodies B, C, *move uniformly in the circumference* ACLGA *of a circle, setting out from* A *at the same time; to find the nature and area of the curve described by their centre of gravity.*

Draw the diameter AOL, O the centre, OG perpendicular to AL; let B, C, be two cotemporary positions



of the bodies; join BC and bisect it in P a point in the curve $APRO$; join OB, take $OE=OP$, draw EF perpendicular to AO, and FH to GO, and join FO. Put $AO=r$, $OP=OE=x$, $AQ=v$, and arc $AB : AC ::$ $1 : n$; then $n+1 : n-1 :: AQ : BQ :: v : BQ = \frac{n-1}{n+1} \times v$. Now the area $OFG = \frac{1}{2}r \times FG$, and $OFH = \frac{1}{2}x \times \sqrt{r^2-x^2}$; hence, $\frac{1}{2}x\sqrt{r^2-x^2} - \frac{1}{2}r \times FG = -HFG$. Now (Art. 46.) $\frac{n-1}{n+1} \times \dot{v}\ (\dot{B}Q) : -\dot{x} :: r : \sqrt{r^2-x^2}$, therefore $\dot{v} = \frac{n+1}{n-1} \times \frac{-r\dot{x}}{\sqrt{r^2-x^2}}$, expressing the equation of the curve. Now (Art. 51.) the fluxion of the area $AOP = \frac{x^2\dot{v}}{2r} = \frac{1}{2} \times \frac{n+1}{n-1} \times \frac{-x^2\dot{x}}{\sqrt{r^2-x^2}}$, whose fluent (Prop. 63.) is $\frac{1}{2} \times \frac{n+1}{n-1} \times$ $(\frac{1}{2}x\sqrt{r^2-x^2} - \frac{1}{2}r \times FG + C) = \frac{1}{2} \times \frac{n+1}{n-1} \times (-$ area $FGH + C)$; but when $AOP=o$, $FGH=AOG$; hence, $-AOG + C = o$, and $C = AOG$; therefore area

$AOP = \frac{1}{2} \times \frac{n+1}{n-1} \times (AOG - FGH) = \frac{1}{2} \times \frac{n+1}{n-1} \times AOHF$; and when $x = 0$, the whole area $APRO = \frac{1}{2} \times \frac{n+1}{n-1} \times AOG$.

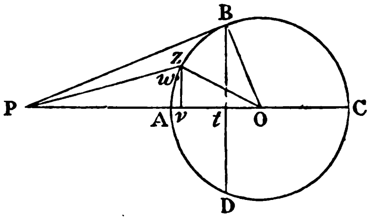
If $n = 3$, the whole area $= AOG$.

If the bodies move in opposite directions, $BQ = \frac{n+1}{n-1}$, and the whole area $= \frac{1}{2} \times \frac{n-1}{n+1} \times AOG$.

Prop. CLVIII.

Let ABCD *be a luminous globe; to find the quantity of light received at* P.

Draw $PAOC$ through the centre O, and BtD perpendicular to it, PB being a tangent at B; take zw



indefinitely small, and draw zv perpendicular to AO, and join OB, Oz. Put $OB = r$, $Ov = x$, $vz = y$, $Bz = z$, $zw = \dot{z}$, $PO = d$, $Ot = a$; then $ad = r^2$, $Pz^2 = d^2 + r^2 - 2dx$. Now as every point is here supposed to give out light equally in all directions, the quantity of light received at P from the annulus generated by zw revolving about AO, varies as that annulus divided by Pz^2, it varies as $\frac{y\dot{z}}{d^2 + r^2 - 2dx} = \frac{r\dot{x}}{d^2 + r^2 - 2dx}$, whose fluent is $\frac{r}{2d} \times -\text{h. l.}\,(d^2 + r^2 - 2dx) + C$; but this vanishes when $x = a$, and the correct fluent is

$\frac{r}{2d} \times$ h. l. $\frac{d^2+r^2-2da}{d^2+r^2-2dx}$; and when $x = r$, we get the whole quantity of light received at P, as $\frac{r}{2d} \times$ h. l. $\frac{d^2-r^2}{d^2+r^2-2dr} = \frac{BO}{2PO} \times$ h. l. $\frac{PB^2}{PA^2}$.

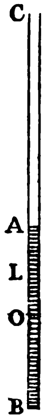
As $\frac{PB^2}{PA^2} = \frac{PO^2-BO^2}{\overline{PO-BO}|^2} = 1 + \frac{2PO}{PO}$ nearly, when PO is very great in respect to BO, the quantity of light in this case varies nearly as $\frac{BO}{2PO} \times$ h. l. $\left(1 + \frac{2BO}{PO}\right) =$ (Art. 103.) $\frac{BO}{2PO} \times \frac{2BO}{PO}$, or as $\frac{BO^2}{PO^2}$.

Prop. CLIX.

Let a small glass tube CB *be partly filled with mercury up to* A; *to find whether it can be filled to such an height, that upon the expansion or contraction of the glass and mercury by an alteration of the temperature of the air, the distance* CO *from the point of suspension at* C *to the centre* O *of occillation may remain the same.*

At any given temperature, let m = the weight of 1 inch in length of the tube, M = that of 1 inch of mercury; and put $a = CB$, $b = AB$ in inches, x = any length CL from C. Then $1 : \dot{x} :: m : m\dot{x}$ the weight or quantity of matter in $\dot{x}$ of the tube, and $1 : \dot{x} :: M : M\dot{x}$ the weight or quantity of matter in $\dot{x}$ of the mercury. Hence (Art. 65.), we proceed thus: the fluent of $mx^2\dot{x}$ = (when $x = a$) $\frac{ma^3}{3}$, and the fluent of $Mx^2\dot{x} = \frac{Mx^3}{3}$, but this must vanish when $x = CL =$

$a-b$; hence, the fluent corrected is $\frac{Ma^3 - M\times(a-b)^3}{3}$.

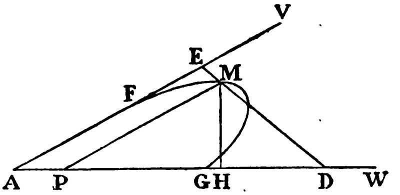


Again, the fluent of $mx\dot{x}$ = (when $x = a$) $\frac{ma^2}{2}$; and the fluent of $Mx\dot{x}$ corrected as before, becomes $\frac{Ma^2 - M\times(a-b)^2}{2}$; hence, $CO = \frac{2}{3} \times \frac{ma^3 + Ma^3 - M\times(a-b)^3}{ma^2 + Ma^2 - M\times(a-b)^2}$. Now if w = weight of the tube, W = that of the mercury; then $m = \frac{w}{a}$, $M = \frac{W}{b}$; and if $\frac{w}{W} = d$, we have, $CO = \frac{2}{3} \times \frac{dba^2 + a^3 - (a-b)^3}{dba + a^2 - (a-b)^2}$. But a change of a and b from a change of temperature, will not change the numerator and denominator in the same ratio; hence, CO will vary with a variation of temperature. There is also another objection to a pendulum thus constructed, that is, that the expansion of the glass and mercury do not take place proportionably in the same time. This compensation pendulum was invented by Mr. Graham; and although it is not perfect, yet a clock with such a pendulum will go about 7 times better, that is, will vary about 7 times less than the same clock with a common pendulum.

LEMMA.

Let AV, AW *be two straight lines,* M *a given point in the straight line* DE, *moveable so as to keep* D, E *in* AW, AV; *to find the curve* FMG *described by* M.

Draw *MP* parallel to *AV*, and *MH* perpendicular to *AW*, and these being given in position, let *MP* :



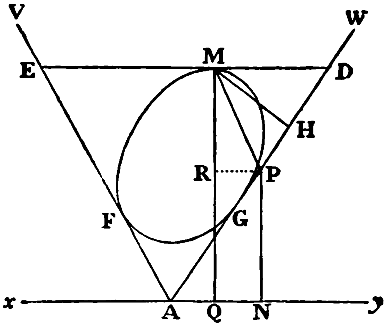
PH :: $a : e$ a constant ratio. Put $AP = x$, $PM = y$, $ME = a$, $MD = b$; then $PH = \frac{ey}{a}$, and $a : b :: x : PD = \frac{bx}{a}$; hence, $HD = \frac{bx - ey}{a}$; also $MH^2 = y^2 - \frac{e^2y^2}{a^2}$, and $b^2 - y^2 + \frac{e^2y^2}{a^2} = HD^2 = \overline{\frac{bx - ey}{a}}\Big|^2$, and $b^2x^2 + a^2y^2 - 2bexy = a^2b^2$ an equation to an ellipse or circle (see Wood's *Algebra.*) The same if *M* lies without *ED*.

Cor. Hence, the construction of an elliptical compass, by making the angle *A* a right angle, and the points *D*, *E* to move in two grooves cut in *AW*, *AV*, a pencil being fixed at *M* as the ruler *DE* turns about.

Prop. CLX.

Let AV, AW, *be two perfectly smooth planes given in position, and the plane* VAW *perpendicular to the horizon* xAy; ED *a rod perfectly smooth at each end,* *lying on the said planes; to find the position of the rod when it will rest.*

By making the rod move, any point M, here supposed the centre of gravity, will (by the Lemma) describe an ellipse FMG. Let M be the highest point of the ellipse, then the centre of gravity of the rod being there, the rod will remain at rest; for as EMD is a tangent to the curve at M, and parallel to the horizon, gravity cannot urge the point M of the rod to move either way. Draw MP parallel to AV, MQ, PN perpendicular to xy, MH to AW, and PR to MQ. Put $EM = a$, $MD = b$, $AP = x$, $PM = y$, and let $PM : PH :: a : e$; then by the Lemma, $b^2x^2 + a^2y^2 - 2bexy = a^2b^2$. Let $s = \text{sin. } PAN$, $t = \text{sin. } MPR$, rad. $= 1$, and these are given from the given position of the planes; then $QR = PN = sx$, $MR = ty$; hence, $sx+ty = MQ$ a maximum, but $y = cx + \sqrt{b^4 - d^2x^2}$ $\left(\text{putting } c = \frac{be}{a^2},\ d^2 = b^2 - \frac{b^2c^2}{a^2}\right)$; hence, $sx + tcx + t\sqrt{b^2 - d^2x^2} =$ (if $s + tc = m$) $mx + t\sqrt{b^2 - d^2x^2} =$ max. and putting it's fluxion $= 0$, we get $x = \frac{mb}{\sqrt{t^2d^4 + m^2d^2}}$; therefore by sim. tri. $a : a+b :: x : AD = \frac{a+b}{a} \times x$ is known.



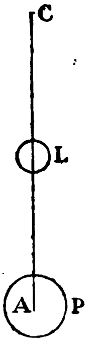
PROP. CLXI.

Let a pendulum consist of a rod CA, *and two balls* P, Q, *of which* P *the larger is fixed at* A, *and the smaller* Q *moveable; given the time* t *of a vibration, to find variation of the place of* Q *corresponding to a small variation of* t.

Let C be the point of suspension; then as we want only to deduce a practical rule, we neglect the weight of the rod, and suppose each ball to be concentrated in it's centre. Let $CQ=x$, $CA=b$; then (Art. 65.) the length of the pendulum $=\frac{Qx^2+Pb^2}{Qx+Pb}$; and by Mechanics, 39,2 inches : $\frac{Qx^2+Pb^2}{Qx+Pb}$ :: $1^{2''}$: $t''^2 = \frac{1}{39{,}2} \times \frac{Qx^2+Pb^2}{Qx+Pb}$. Now as the increments of t and x are supposed to be very small, they will be nearly as their fluxions; hence, $2t\dot{t} = \frac{Q^2x^2\dot{x}+2PQbx\dot{x}-PQb^2\dot{x}}{39{,}2\times\overline{Qx+Pb}|^2}$,

and $\dot{x} = \frac{39{,}2\times 2t\times\overline{Qx+Pb}|^2}{Q^2x^2+2PQbx-PQb^2}\times\dot{t}$.

Let $b=158$ inches, $x=126{,}9$, $P=20$lbs.; $Q=1$lb., then $t=2''$; and suppose the clock by the variation of x



to lose 15″ in 24 hours, or 86400″; then 86400″ : 2″ :: 15″ : 0″,000347 the time lost in 2″, or $\dot{t}$; hence, $\dot{x}$ = 1,84 inches, the corresponding variation of x, or of the place of Q.

Clocks whose pendulums have large bobs, are corrected in this manner, as it can be done more accurately, the scale of variation being longer.

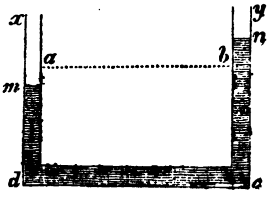
LEMMA.

If l = the length of a pendulum describing a cycloid whose axis is perpendicular to the horizon; then by the property of the cycloid, the accelerative force at the arc x of distance from the lowest point : force of gravity :: $x : l$; and this we may apply in all oscillations and vibrations when the acceleration varies as the space to be described from the point at rest, that being the law in the cycloid.

PROP. CLXII.

Let a tube xdcy *of uniform size, the parts* xd, yc *being upright, and filled with a fluid up to* ab, *and then depressed from* a *to* m *so as to force the fluid to rise on the other side to* n; *then on removing the force, to find the time in which the fluid will oscillate.*

Put $x = am = bn$, l = length of the canal $adcb$; then the fluid at n standing at the altitude $2x$ above m, $2x$ =

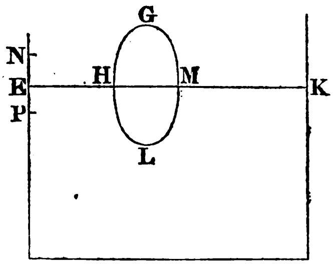


moving force; hence $\frac{2x}{l}$ = accelerating force; and by the Lemma, $\frac{2x}{l}$: gravity (1) :: x : $\frac{l}{2}$ the required length of the pendulum.

Hence, if this oscillation be analogous to the motion of the waves of the sea, **Newton** concludes that the waves move through a space equal to their breadth, nearly in the time in which a pendulum whose length is half the breadth of the wave, makes a vibration.

Prop. CLXIII.

Let GHLM *be a body* (B) *at rest in a vessel of water whose surface is* EHMK; *depress the body through a very small space* EP *so that the water may rise to* N, *and then let the body go; to find the time of an oscillation.*



Let a = section HM of the body, b = section EK, M = the magnitude of the body, V = part HLM, $EP = x$; then as the part of the body depressed = the water elevated, $EN \times (b-a) = ax$, and $EN = \frac{ax}{b-a}$, therefore $NP = \frac{ax}{b-a} + x = \frac{bx}{b-a}$, and $\frac{abx}{b-a}$ = the water displaced, or the moving force to drive up the body. But a quantity of water V acting upwards against the body, keeps B at rest, or is equivalent in effect to B; to find therefore the equivalent effect of $\frac{abx}{b-a}$, we say $V : \frac{abx}{b-a} :: B : \frac{B}{V} \times \frac{abx}{a-b}$ which therefore represents the moving force acting upwards against B;

hence, the accelerating force $= \frac{1}{V} \times \frac{abx}{b-a}$; and this varying as x the space from the equilibrium point, we have (by Lem.) $\frac{1}{V} \times \frac{abx}{b-a} : 1$ (Gr.) $:: x : V \times \frac{b-a}{ab} = \frac{V}{a} - \frac{V}{b}$, the length ($l$) of the pendulum required.

If b be indefinitely great in respect to a, $l = \frac{V}{a}$. Let $V =$ a cylinder whose base is a, and altitude h, then $h = \frac{V}{a} = l$.

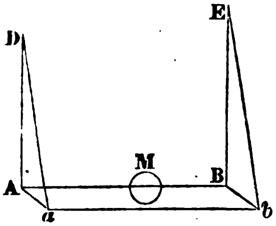
To find the weight of a ship's loading, observe the length l of a pendulum which vibrates in the time a ship is found to make one oscillation, and $a =$ the area of the section of the ship at the surface of the water, and we get V (la) the content of the part immersed. Hence, knowing the weight of a cubic foot of water, we get the weight of the whole ship including the loading; from which subtract the weight of the ship, and we get the weight of the loading.

If the body be a cylinder whose axis is perpendicular to the surface of the fluid, the proposition is true, to whatever depth we depress the body, for then a remains constant. Now if $h =$ the length of the part immersed, $V = h \times a$; hence, $h = \frac{V}{a} = l$, when b is indefinitely great.

PROP. CLXIV.

The line AB *having a weight* M *upon it, is supported by two strings* AD, BE, *and made to vibrate through a very small arc; to find the length of the pendulum.*

Put $Aa = Bb = x$ the arc described, $AB = c$, $AM = a$, $BM = b$, $AD = m$, $BE = n$; then the weight

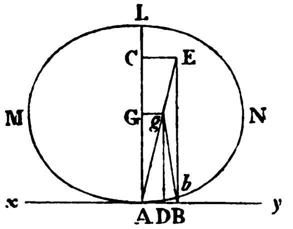


of M at $A = M \times \frac{b}{c}$, and at B it $= M \times \frac{a}{c}$. Now the accelerative force of a pendulum varies as the arc divided by radius, the arc being very small; apply this to $M \times \frac{b}{c}$ and $M \times \frac{a}{c}$, and we get the motions generated as $M \times \frac{b}{c} \times \frac{x}{m}$ and $M \times \frac{a}{c} \times \frac{x}{n}$ and the whole moving force $= M \left(\frac{b}{c} \times \frac{x}{m} + \frac{a}{c} \times \frac{x}{n}\right)$; divide this by M, and we get the accelerative force $= \frac{x}{c} \times \left(\frac{b}{m} + \frac{a}{n}\right)$, which varying as x, we have $\frac{x}{c} \times \left(\frac{b}{m} + \frac{a}{n}\right) : 1$ (Gr.) $:: x : \frac{mnc}{bn + am}$ the length of the pendulum required.

Prop. CLXV.

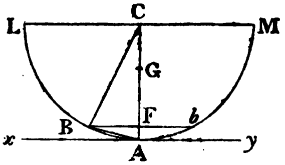
Let a body MN *rest on an horizontal plane* xy *at* A, *and rolling it through a very small arc* Ab, *let it return back and oscillate; to find the length of a pendulum which shall oscillate in the same time.*

Draw AL perpendicular to xy, and let G be the centre of gravity, C the centre of a circle of curvature to the arc Ab; take $AB = Ab$, so that the body rolling from *A* to *B*, *G* may move to *g* and *C* to *E* in lines parallel to *AB*, the two points turning about *A*. Now



when a circle rolls in a straight line, it's centre must continue in a perpendicular to that line from the point of contact; *C* therefore may be considered as the centre of a circle coinciding with *Ab*, and which comes to *E* when *b* coincides with *B*, *BE* being perpendicular to *xy*; hence, $CE = AB$; also, *E*, *g*, *A*, must be in the same straight line, the velocities, *Gg*, *CE*, being as *AG*, *AC*; and the centre of gravity now at *g* not being supported, if *gD* be perpendicular to *AB*, the body will (by Mechanics), oscillate back by a force which is to gravity as sin. *DgB* to rad. or as $DB : Bg$ or *AG*. Put $AG = a$, $AC = b$, $AB = x$; then $b : a :: x : \frac{ax}{b} = Gg = AD$; hence, $BD = x - \frac{ax}{b} = \frac{b-a}{b} \times x$; also, let $M =$ quantity of matter in the body, $r =$ the distance of any particle *p* from *A*, the body being supposed to be orthographically projected on the plane *AMLN* passing through the centre of gravity, and in which the body oscillates. Now $gB : DB :: M :$ that part of gravity which tends to bring the body back $= \frac{M \times DB}{gB} = M \times \frac{b-a}{ba} \times x$ the moving force. But (Art. 60.) $\int r^2 p$ represents the inertia of the whole body; and to find the quantity of matter *m* which placed at *G* shall have the same inertia, we have by the same Art. $\int r^2 p = a^2 m$, and $m = \frac{\int r^2 p}{a^2}$. Hence, we may conceive the moving force $M \times \frac{b-a}{ba} \times x$ as having to move a body $\frac{\int r^2 p}{a^2}$; therefore the accelerating force $= M \times \frac{b-a}{ba} \times x \times \frac{a^2}{\int r^2 p}$, which varies as x; hence, by the Lemma $M \times \frac{b-a}{ab} \times x \times \frac{a^2}{\int s^2 p} : 1$ (Gr.) :: $Gg = \frac{ax}{b}$ (the indefinitely small space through which the body has oscillated) : the length of the pendulum $= \frac{\int r^2 p}{M \times (b-a)}$.

Ex. Let *LAN* be the arc of a semicircle whose centre is *C*; from any point *B* draw *BFb* perpendicular

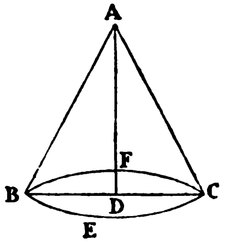


to *CA*; put $CB = r$, $AF = x$, arc $AF = z$, arc $AL = c$; then chord $AB = \sqrt{2rx}$, and $BF = \sqrt{2rx - x^2}$; also (Art. 46.) $\dot{z} = \frac{r\dot{x}}{\sqrt{2rx - x^2}}$; hence, $\int r^2 p = \int 2rx \times 2\dot{z} = \int \frac{4r^2 x\dot{x}}{\sqrt{2rx - x^2}} = 4r^2 \times (z - \sqrt{2rx - x^2}) =$ Pr. 69. (when $x = r$) $4r^2 \times (c - r)$. Also (Art. 58. Ex. 5.) $AG = r - \frac{r^2}{c}$; and $M = 2c$; hence, the length of the pendulum $=$

$$\frac{4r^2 \times (c-r)}{2c \times \left(r - \frac{r^2}{c}\right)} = 2r.$$

PROP. CLXVI.

***To find the time t'' in which a pendulum will describe a conical surface* CABEF.**



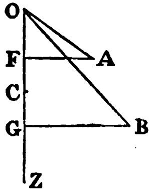
Let *AD* be the altitude of the cone, v the velocity of the body *B* in the circumference *BECF*; then (Art. 202.) $\frac{v^2}{DB}$ = centrifugal force of *B*, the force of gravity being $32\frac{1}{6}=m$; and these two forces acting on *B* in the directions *DB* and parallel to *AD*, if we consider *DB*, *AD* as two levers, the effect of the former to urge *B* upwards is $\frac{v^2}{DB} \times AD$, and of the latter to urge *B* downwards is $m \times BD$; make these equal and $v = BD \times \sqrt{\frac{m}{AD}}$. If therefore $p = 6{,}2838$, then $p \times BD$ = circumference *BECFB*; hence, $BD \times \sqrt{\frac{m}{AD}} : p \times BD :: 1'' : t'' = p\sqrt{\frac{AD}{m}}$.

COR. The time T'' down $2AD = \sqrt{\frac{4AD}{m}}$; hence, $t'' : T'' :: p\sqrt{\frac{AD}{m}} : p\sqrt{\frac{4AD}{m}} :: p : 2$ :: circumference : diameter of a circle.

PROP. CLXVII.

Let A, B, *be two indefinitely small bodies, attached to* *the point* O, *so as to describe in the same time their respective conical surfaces; to find that time;* O, A, B *lying in the same vertical plane.*

Draw OZ perpendicular to the horizon, AF, BG perpendicular to it; and let c = the velocity about OZ at the distance unity; then $1 : AF :: c : c \times AF$ = the



velocity of A; therefore (Art. 202.) $c^2 \times FA \times A$ = the centrifugal force of A, and it's efficacy as before to urge the system upwards is $c^2 \times FA \times FO \times A$; for the like reason, the efficacy of B is $c^2 \times GB \times GO \times B$ is the efficacy of B; also the efficacy of gravity to turn A and B in the opposite direction is $m \times (A \times FA + B \times GB)$; hence, $c^2 \times FA \times FO \times A + c^2 \times GB \times GO \times B = m \times (A \times FA + B \times GB)$, and $c = \sqrt{\frac{m \times (A \times FA + B \times GB)}{FA \times FO \times A + GB \times GO \times B}}$; hence, $c : p :: 1'' :$

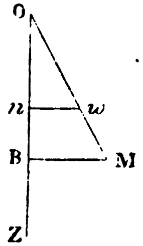
$$t'' = \frac{p}{c} = \frac{p}{\sqrt{m}} \times \sqrt{\frac{FA \times FO \times A + GB \times GO \times B}{A \times FA + B \times GB}}$$

the time of a revolution.

In general under the condition stated in the proposition, if z represent any plane body, $\dot{z}$ any particle whose perpendicular distance from OZ is x, and from O in the direction ZO is y, then $t = \frac{p}{\sqrt{m}} \times \sqrt{\frac{\text{flu. } xy\dot{z}}{\text{flu. } x\dot{z}}}$, where the part on the opposite side of OZ must be considered as negative.

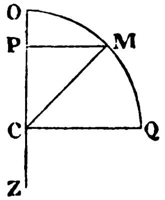
To find the height h of a cone described by a simple pendulum in the same time, by the last Prop. $\frac{p}{\sqrt{m}} \times$. $\sqrt{h} = \frac{p}{\sqrt{m}} \times \sqrt{\frac{\text{flu.}\, xy\dot{z}}{\text{flu.}\, x\dot{z}}}$; and h $= \frac{\text{flu.}\, xy\dot{z}}{\text{flu.}\, x\dot{z}}$.

Ex. 1. Let *OM* be a uniform slender rod describing a conical surface; draw *MB*, *wn* perpendicular to *OZ*;



put $Ow = z$, $x = nw$, $y = On$, s = sin. c = cos. of angle *ZOM*; then $x = sz$, $y = cz$, and the fluent of $xy\dot{z}$ = fluent of $scz^2\dot{z} = \frac{1}{3}scz^3$, and fluent $x\dot{z}$ = fluent $sz\dot{z} = \frac{1}{2}sz^2$; hence, $h = \frac{2}{3}cz = \frac{2}{3}OB$. And when the angle *O* is indefinitely small, h = the length of a simple pendulum describing a circular arc.

Ex. 2. Let *OMQ* be a quadrantal arc, whose centre *C* lies in *OZ*, draw *MP* perpendicular to *OZ*; put



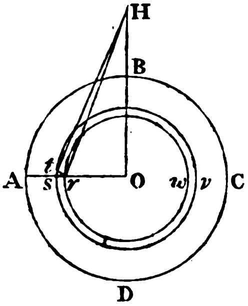
$r = CM$, $x = PM$, $y = PO$, $Z = OM$; then (Art. 46.) $x\dot{z} = r\dot{y}$; hence, fluent $xy\dot{z}$ = fluent $ry\dot{y} = \frac{1}{2}ry^2$, and fluent $x\dot{z}$ = fluent $r\dot{y} = ry$; and $h = \frac{1}{2}y$.

Hence, whatever r may be, whilst *OP* continues the same, the time of revolution is the same; which is the case of a simple pendulum when the height of the cone remains the same. And the two revolutions are equal when the altitude of the cone $= \frac{1}{2} OP$.

PROP. CLXVIII.

If a candle be elevated above the centre of a circular table; to find the light received upon it.

Let $ABCD$ be the table, O the centre, OH perpendicular to the table, H the place of the candle, $rsvw$ a



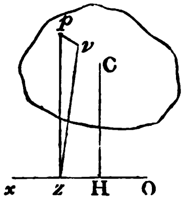
concentric annulus, and rt perpendicular to sH. Put $p = 6{,}283$ &c. $x = Hr$, $a = HO$, then $Or^2 = x^2 - a^2$, and $x\dot{x} = Or \times O\dot{r} = Or \times rs$. Now no more light falls on rs than what falls on tr, and therefore it varies as $\frac{tr}{Hr^2}$; hence, the quantity of light fallen on the annulus varies as $\frac{p \times Or \times tr}{Hr^2} =$ (as $Hr : HO :: rs : tr$) $\frac{p \times HO \times Or \times sr}{Hr^3} = \frac{pax\dot{x}}{x^3} = \frac{pa\dot{x}}{x^2}$ whose fluent is $-\frac{a}{x} + C$ the light fallen on the area whose radius is rO, which vanishes when $x = a$, and the correct fluent is $1 - \frac{a}{x} = 1 - \frac{HO}{HA}$.

If rs be indefinitely small and given, and Or be given; then the quantity of light on the annulus is as $\frac{HO}{Hr^3}$; and to find the height OH when the light received on the annulus is a maximum; put $Or = a$, $OH = x$; then $\frac{x}{\overline{a^2+x^2}|^{\frac{3}{2}}} =$ max. whose fluxion made $= 0$, we get $x = \frac{a}{\sqrt{2}}$.

Prop. CLXIX.

If a body revolve about an axis x O, *the centrifugal force is the same as if all the matter were collected into the centre of gravity* C.

Let a plane of the body pass through C and xO; p a particle of the body, pv perpendicular to that plane,

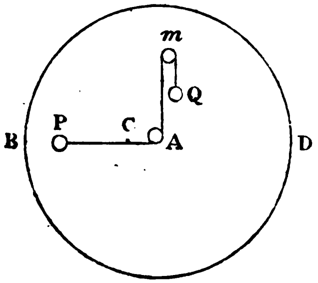


and CH, pz, vz to xO. Then as the centrifugal force or $p \propto pz$, resolve this force into $p \times vp$, $p \times zv$; then by Mechanics, the sum of all the $p \times vp = 0$, and the sum of all the $p \times zv =$ body $\times HC$; that is, the whole effect of the centrifugal force = the centrifugal force of the whole body placed at C.

Prop. CLXX.

Let BD *be an horizontal table moveable about the centre* A; P *a body lying on it and connected with a body* Q *by a string going from it's centre of gravity parallel to the table, passing under a pulley at* A *and* *over another at* m *vertical to* A, Q *hanging freely down; and let the table be turned about* A *and carry* P *with it, so that the centrifugal force of* P *may draw* Q *up,* P *being constrained to move in the line* ACP; *to find the time of describing* CP, P *setting out from* C.

At the distance 1, let c = circular velocity of the table, $a = AC$, $x = AP$, v=velocity of P, the matter in



P being conceived to be concentrated in it's centre of gravity, t=time of describing CP, $m=32\frac{1}{6}$ feet. Then (Art. 202.) gravity being represented by m, the centrifugal force of $P = \frac{c^2x^2}{x} \times P = c^2 x P$; therefore the moving force $= c^2 x P - mQ$, and the accelerative force $= \frac{c^2 x P - m Q}{P+Q}$; and (Art. 82.) $v\dot{v} = \frac{c^2 x P - m Q}{P+Q} \times \dot{x}$, whose fluent is $\frac{v^2}{2} = \frac{c^2 P}{P+Q} \times \frac{x^2}{2} - \frac{mQ}{P+Q} \times x + C$; but when $v = 0$, $x = a$, and putting $c^2 - \frac{c^2 P}{P+Q} \times a^2 + \frac{2mQ}{P+Q} \times a = b$, $\frac{c^2 P}{P+Q} = d$, $\frac{2mQ}{P+Q} = e$, the correct fluent is $v^2 = dx^2 - ex + b$; hence, $\dot{t} = \frac{\dot{x}}{\sqrt{dx^2 - ex + b}} =$

$\frac{1}{\sqrt{d}} \times \frac{\dot{x}}{\sqrt{x^2 - \frac{e}{d}x + \frac{b}{d}}}$. Put $x - \frac{e}{2d} = y$, then $\dot{x} = \dot{y}$; and (putting $\frac{b}{d} - \frac{e^2}{4d^2} = n^2$), we get $\dot{t} = \frac{1}{\sqrt{d}} \times \frac{\dot{y}}{\sqrt{y^2 + n^2}}$, and $t = \frac{1}{\sqrt{d}} \times \text{h. l.}\ (y + \sqrt{y^2 + n^2}) + C$; but when $t = 0$, $x = a$, and therefore $y = a - \frac{e}{2d} = g$; hence, the correct fluent is $t = \frac{1}{\sqrt{d}} \times \text{h. l.}\ \frac{y + \sqrt{y^2 + n^2}}{g + \sqrt{g^2 + n^2}}$.

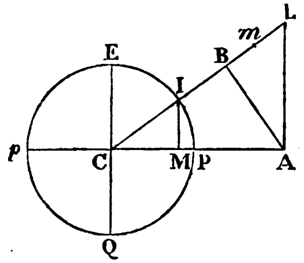
When $c^2 x P - m Q = 0$, or $x = \frac{mQ}{c^2 P}$, the accelerative force vanishes; if therefore P were to be set off at that distance, it would remain at rest.

On this principle the *whirling table* is constructed. Instead of a complete table, different arms are put on, and by means of different wheels, P is made to revolve at different distances in different times, and different weights Q may represent different attractive forces. Thus the periodic times, distances and forces may be adjusted as we please. If they be adjusted to the case of the planets, the experiment proves the law of gravity to be in the inverse square of the distance.

PROP. CLXXI.

To find the altitude of the highest cylinder that can be raised on any part of the earth's surface.

Let C be the centre of the earth, P, p it's poles, EQ the equator, CL the cylinder in the direction CI of the radius. Draw IM, LA perpendicular to CPA, and AB to CL. Now when the centrifugal force at L in opposition to gravity be equal to gravity at that point, the cylinder can be raised no higher, since at any greater altitude, the centrifugal force would be greater



than gravity, and therefore the parts would be thrown off. Put $r = CE = CI$, $b = IM$, $d = CM$, $x = CL$, $c = \frac{1}{289}$ the centrifugal force at E, gravity being unity; then by sim. tri. $r : b :: x : AL = \frac{bx}{r}$; hence (Art. 202.) $r : \frac{bx}{r} :: c : \frac{cbx}{r^2}$ the centrifugal force at L in the direction AL, L revolving about A; and $AL : BL$, or $r : b :: \frac{cbx}{r^2} : \frac{cb^2x}{r^3}$ the centrifugal force in the direction IL. Also, $\frac{1}{r^2} : \frac{1}{x^2} :: 1$ (gravity) $: \frac{r^2}{x^2}$ the gravity at L; hence, $\frac{cb^2x}{r^3} = \frac{r^2}{x^2}$, and $x = \frac{r^{\frac{5}{3}}}{c^{\frac{1}{3}}b^{\frac{2}{3}}}$, and $\frac{r^{\frac{5}{3}}}{c^{\frac{1}{3}}b^{\frac{2}{3}}} - r = IL$ the altitude of the cylinder required.

To find the weight w of this cylinder, put $x = Cm$, $n =$ the weight of a cubic foot at the surface; then the whole force at m towards $C = \frac{r^2}{x^2} - \frac{cb^2x}{r^3}$, which at I becomes $1 - \frac{cb^2}{r^2}$; hence, $1 - \frac{cb^2}{r^2} : \frac{r^2}{x^2} - \frac{cb^2x}{r^3} :: n :$

$\frac{er^2}{x^2} - \frac{cb^2ex}{r^3}$ (putting $e = \frac{nr^2}{r^2 - cb^2}$) the weight of a cubic foot at m; hence, $\frac{er^2\dot{x}}{x^2} - \frac{cb^2ex\dot{x}}{r^3} = \dot{w}$, and $w = -\frac{er^2}{x} - \frac{cb^2ex^2}{2r^3} + C$; but when $w = 0$, $x = r$, and the correct fluent is $w = er + \frac{ecb^2}{2r} - \frac{er^2}{x} - \frac{ecb^2x^2}{2r^3}$; and putting $\frac{r^{\frac{5}{3}}}{c^{\frac{1}{3}}b^{\frac{2}{3}}}$ for x, we get $w = er + \frac{eb^2c}{2r} - ec^{\frac{1}{3}}b^{\frac{2}{3}}r^{\frac{1}{3}} - \frac{1}{2}ec^{\frac{1}{3}}b^{\frac{2}{3}}r^{\frac{1}{3}}$.

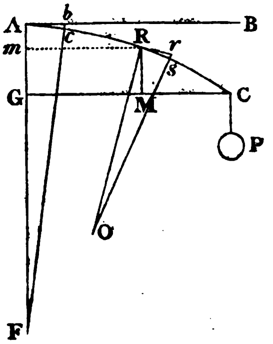
At the pole, $b = 0$, $w = er = nr$, and x becomes infinite. At the equator, $b = r$, and $w = nr \times \left(\frac{2 + c}{2} - \frac{3}{2}c^{\frac{1}{3}}\right)$.

PROP. CLXXII.

If a perfectly flexible horizontal rod AB *be bent through a very small space into the curve* AC *by a weight* P *hanging from it's extremity; to find the nature of the curve* AC.

Draw AF perpendicular to AB, and CG to AF; take Ac, Rs two equal indefinitely small parts of the curve, F, O, the centers of curvature to Ac, Rs, and join cF, RO, sO, and draw cb perpendicular to AB, and sr to the tangent Rr, and RM perpendicular to CG. Put $CM = x$, $MR = y$, then by the property of the lever, the effect of P at A : effect at R :: $CG : CM$; but these effects are as bc, rs, hence, $CG : CM :: bc : rs :: \frac{Ac^2}{AF} : \frac{Rs^2}{RO} :: RO : AF$, and $CM \times RO = CG \times$

AF = a^2 a constant quantity; hence, $\frac{x \times \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}} =$

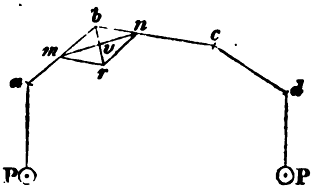


a^2, and $x\dot{x} = \frac{-a^2\dot{x}^2\ddot{y}}{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}}}$. Put $\dot{y} = \frac{\dot{x}^2}{\dot{v}}$, then $\ddot{y} = \frac{-\dot{x}^2\ddot{v}}{\dot{v}^2}$; hence, $x\dot{x} = \frac{a^2\dot{x}^4\dot{v}\ddot{v}}{\overline{\dot{x}^2\dot{v}^2 + \dot{x}^4}|^{\frac{3}{2}}} = \frac{a^2\dot{x}\dot{v}\ddot{v}}{\overline{\dot{v}^2+\dot{x}^2}|^{\frac{3}{2}}}$, whose fluent is $\frac{1}{2}x^2 = \frac{-a^2\dot{x}}{\sqrt{\dot{v}^2+\dot{x}^2}}$; but $\dot{v}^2 = \frac{\dot{x}^4}{\dot{y}^2}$; hence, $\frac{1}{2}x^2 = \frac{-a^2\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}$, and $\dot{y} = \frac{x^2\dot{x}}{\sqrt{4a^2 - x^4}}$ the equation of the curve.

Prop. CLXXIII.

If two equal weights P, P, *be connected by a string passing over any number of tacks,* a, b, c, d; *to find the pressure against each tack.*

As the tension of the string is every where the same, let *bm* = *bn* represent that tension, or *P*, and complete the parallelogram *mbnr*, and let the diagonals intersect at *v*, then *br* is the compound force arising from the tension, or the pressure on *b*; hence, the pressure on the tack b : P :: br, or $2bv$, : bm :: twice cos. of half



the angle mbn : radius.

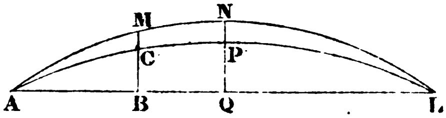
If the number of tacks be increased and their distances diminished, so that they may approach a curve as their limit, bv becomes it's sagitta, which, as the increment of the curve is given, varies inversely as the radius of curvature. Hence, a string going over a pulley, presses equally on every point.

Hence, a musical string during it's curvilinear form, endeavours to restore itself in a direction perpendicular to every point, by a force which is inversely as the radius of curvature.

Prop. CLXXIV.

To find the motion of a musical string AMNL.

Let the string be fixed at A, L, AQL it's quiescent position, $AMNL$ it's extreme position, $ACPL$ any



other position; bisect AL in Q, and draw QPN, BCM perpendicular to AL. Put $AQL=a$, $NQ=b$, $AB=x$, $BC=y$, $AC=z$, $R=$radius of curvature at C, $r=$that at N, $w=$weight of the string, $l=$it's tension, $m = 16\frac{1}{12}$ feet, $c = 3,14159$ &c. Now as every part of the string comes into the position AQL at the same time, every point C must be urged towards B by a force which is as the distance CB (Art. 83. Ex. 2.);

and by the last Proposition, it is urged by a force in a direction perpendicular to the curve, or, as the vibrations are supposed very small, in the direction CB, and is inversely as R, or inversely as $\frac{\dot{y}\dot{z}}{\ddot{x}}$, $\dot{z}$ being constant; we have therefore $y : b :: r : \frac{\dot{y}\dot{z}}{\ddot{x}}$, and $y\dot{y}\dot{z} = br\ddot{x}$, and the fluent is $br\dot{x} = \frac{1}{2}y^2\dot{z} + C$; but at N, $y = b$, $\dot{x} = \dot{z}$; hence, the correct fluent is $br\dot{x} - br\dot{z} = \frac{1}{2}(y^2 - b^2)\dot{z}$, and $\dot{x} = \frac{\frac{1}{2}(y^2 - b^2) + br}{br} \times \dot{z} = \frac{\frac{1}{2}(y^2 - b^2) + br}{br} \times \sqrt{\dot{x}^2 + \dot{y}^2}$; hence, $\dot{x} = \frac{(2br + y^2 - b^2) \times \dot{y}}{\sqrt{4b^3r - 4bry^2 - y^4 - b^4 + 2b^2y^2}} =$ nearly (as r is very great in respect to y and b) $\sqrt{br} \times \frac{\dot{y}}{\sqrt{b^2 - y^2}}$, and $x = (A)$ a circular arc whose sine is $\frac{y}{b}$ and radius $= 1$, and when $y = b$, $x = \frac{1}{2}a$; hence, $\frac{1}{2}a = \sqrt{rb} \times \frac{c}{2}$, and $r = \frac{a^2}{c^2 b}$. Now let us put $x = PQ$, v = velocity at P, then the radius of curvature at $P = \frac{a^2}{c^2 x}$ by our last step; and by the last Prop. the motive force of $P : l :: \dot{z} : \frac{a^2}{c^2 x}$, and that force $= \frac{lc^2 x\dot{z}}{a^2}$; but $a : w :: \dot{z} : \frac{w\dot{z}}{a}$ the weight of $\dot{z}$; hence, the accelerative force of $P = \frac{lc^2 x}{wa}$, and (Art. 82.) $v\dot{v} = -\frac{2mlc^2}{wa} \times x\dot{x}$, therefore $v^2 = \frac{-2mlc^2}{wa} \times (x^2 + C)$; but when

$v = 0$, $x = b$, and $v^2 = \frac{2mlc^2}{wa} \times (b^2 - x^2)$, and $v = \sqrt{\frac{2ml}{wa}} \times c \times \sqrt{b^2 - x^2}$, and when $x = 0$, $v = bc\sqrt{\frac{2ml}{wa}}$ the velocity at Q.

Lastly, if t = time through NP, $\dot{t} = \frac{-\dot{x}}{v} = \frac{1}{c\sqrt{\frac{2ml}{wa}}} \times \frac{-\dot{x}}{\sqrt{b^2 - x^2}}$, and $t = \frac{1}{c\sqrt{\frac{2ml}{wa}}} \times$ (− circ. arc (A) whose sine is $\frac{x}{b}$, rad. = 1 + C); but when $t = 0$, $x = b$, and $t = \frac{1}{c\sqrt{\frac{2ml}{wa}}} \times$ (− A + quadrantal arc) $= \frac{1}{c\sqrt{\frac{2ml}{wa}}} \times q$, q being an arc whose cosine is $\frac{x}{b}$, and when $x = 0$, $q = \frac{1}{2}c$, and $t = \frac{1}{2}\sqrt{\frac{wa}{2ml}}$.

PROP. CLXXV.

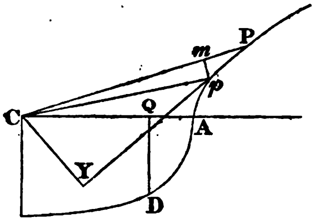
To find the equation of the elastic curve.

The property of this curve, is, that the ordinate varies as the curvature. Put x = abscissa, y = ordinate, z = curve; then if $\dot{z}$ be constant, the radius of curvature $= \frac{\dot{y}\dot{z}}{\ddot{x}}$; but the curvature varies inversely as the radius; hence, $y \propto \frac{\ddot{x}}{\dot{y}\dot{z}}$, or $\frac{2y}{a^2} = \frac{\ddot{x}}{\dot{y}\dot{z}}$, and $2y\dot{y}\dot{z} = a^2\ddot{x}$, and $\dot{z}$ being constant, the fluent is $y^2\dot{z} = a^2\dot{x} + m^2\dot{z}$,

$m^2\dot{z}$ being a constant quantity to correct the fluent; hence, $a^2\dot{x} = (y^2 - m^2) \times \dot{z} = (y^2 - m^2) \times \sqrt{\dot{x}^2 + \dot{y}^2}$, and $\dot{x} = \pm \dfrac{(y^2 - m^2) \times \dot{y}}{\sqrt{a^4 - \overline{y^2 - m^2}|^2}}$. If $m = 0$, $\dot{x} = \dfrac{\pm y^2\dot{y}}{\sqrt{a^4 - y^4}}$.

By the last Prop. it appears that the musical string approaches to the elastic curve as it's limit.

The curve $\dfrac{y^2\dot{y}}{\sqrt{a^4 - y^4}}$ may be constructed by the rectification of the equilateral hyperbola. Let *C* be the centre of an equilateral hyperbola *PA*, *A* the vertex, *CY* perpendicular to the tangent *PY*; and take



CQ = *CY*, the ordinate *QD* = *PY* = *PA*, and *D* is a point in the curve. Put $x = QD$, $y = CQ$, $a = CA$, $AP = z$, $PY = w$, $CP = v$, then $v = \dfrac{a^2}{y}$, $w = \sqrt{v^2 - y^2} = \dfrac{\sqrt{a^4 - y^4}}{y}$, $\dot{w} = \dfrac{-y^4\dot{y} - a^4\dot{y}}{y^2\sqrt{a^4 - y^4}}$; but by sim. tri. *Pmp*, *PCY*, *Pp* being indefinitely small, $\dot{z} : \dot{v} :: v : w :: a^2 : \sqrt{a^2 - y^2}$, and as $v = \dfrac{a^2}{y}$, $\dot{v} = \dfrac{-a^2\dot{y}}{y^2}$, we have $\dot{z} = \dfrac{-a^4\dot{y}}{y^2\sqrt{a^4 - y^4}}$; but $x = w - z$, and $\dot{x} = \dot{w} - \dot{z} = \dfrac{-y^2\dot{y}}{\sqrt{a^4 - y^4}}$ the equation to be constructed.

In general the curve may be constructed by the hyperbola and ellipse; see *Maclaurin's* Fluxions, Art. 928.

PROP. CLXXVI.

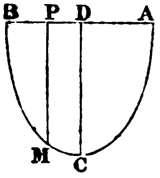
To find the fluent of $\frac{v^2 \dot{v}}{\sqrt{v^4-1}}$.

The construction of the last figure remaining, let $CA = 1$, $CP = v$; then $CY = \frac{1}{v}$, $PY = \sqrt{v^2 - \frac{1}{v^2}} = \sqrt{\frac{v^4-1}{v}}$; and by sim. tri. $\dot{z} : \dot{v} :: v : \frac{\sqrt{v^4-1}}{v}$, and $\dot{z} = \frac{v^2 \dot{v}}{\sqrt{v^4-1}}$; the required fluent is therefore the arc AP generated whilst v flows from AC or 1.

PROP. CLXXVII.

Let ACB *be a curve supporting a fluid; to find it's nature.*

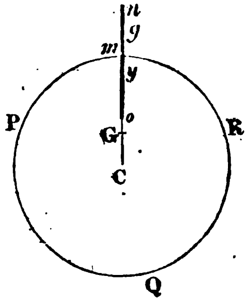
Let $DB = b$, $DP = x$, $PM = y$, $CM = z$; then as the pressure of the fluid is as it's depth, and equal in all directions, the pressure perpendicular to the curve at M varies as y; this therefore is as the curvature, or inversely as the radius $\frac{\dot{y}\dot{z}}{\ddot{x}}$, $\dot{z}$ being constant; assume therefore $\frac{2y}{a^2} = \frac{\ddot{x}}{\dot{y}\dot{z}}$; hence, this is the same equation as in Prop. 175. and $\dot{x} = \frac{(y^2 - m^2) \times \dot{y}}{\sqrt{a^4 - \overline{y^2 - m^2}^2}}$.



Prop. CLXXVIII.

Let PQRm *be a great circle of a sphere perpendicular to its axis of rotation, passing through the centre* C; om *a slender cylindrical part of the radius* Cm *in a direction passing through* C; *and whilst the globe is revolving, let* om *be moved in the direction of the radius into the position* mn; *to find how much the time of rotation will be altered.*

Let M be the quantity of matter in the sphere, supposed of uniform density, m the quantity in the cylinder



om or mn, G the centre of gyration of the globe, y, g, those of the cylinders om, mn; then (Art. 61. Ex. 1.) $Cy = \sqrt{\frac{Cm^3 - Co^3}{3mo}}$, $Cg = \sqrt{\frac{Cn^3 - Cm^3}{3mn}}$. Now the taking away of the cylinder mo, is, in respect to its inertia, the same as taking away a quantity of matter m placed at y; and adding the cylinder mn, is the same as placing m at g. Now to find the quantity of matter (q) at G which is equivalent in its inertia to m at y, we have (Art. 60.) $q \times CG^2 = m \times Cy^2$, and $q = \frac{m \times Cy^2}{CG^2}$; in like manner, the quantity of matter at G equivalent in its inertia to m at g, is $\frac{m \times Cg^2}{CG^2}$. The whole quantity of matter therefore to be added to M

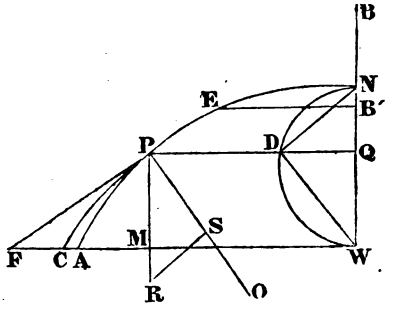
placed at *G*, which shall be equal in its inertia to the removal of the cylinder *om* into the position *mn*, is $\frac{m}{CG^2} \times (Cg^2 - Cy^2)$. Hence the inertia of the globe at first : inertia afterwards :: $M : M + \frac{m}{CG^2} \times (Cg^2 - Cy^2)$; and as the velocity must be diminished in the ratio of this increase of inertia, the periodical time must be increased in that ratio.

Hence, such an alteration of the Earth from subterraneous forces, may cause an alteration in its time of rotation, though probably too small to be sensible.

PROP. CLXXIX.

Let a body descend down any curve NA; *to find the pressure at any point* P.

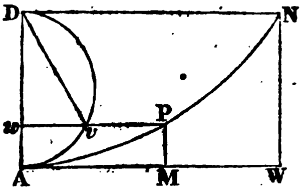
Let *BNW*, *PMR*, be perpendicular to the horizon, *PDQ*, *AMW*, parallel to it, *PO* the radius of curvature at *P*, and let $PR = 1$ represent the force of gravity,



which resolve into *RS* perpendicular to *PO*, and *PS*. Put $MA = x$, $PM = y$, $AP = z$, $NW = a$ the diameter of the semi-circle *NDW*, $PO = r$, $BN = n$ the space through which a body must fall to acquire the velocity at *N* in the curve, $m = 16\frac{1}{12}$ feet, v = velocity of a body revolving in a circle at *N* about the center *W*, V = the velocity of the body in the circle of curvature at *P*, or in the curve at P. Now (Art. 210.) a body must fall down $\frac{1}{2}a$ to acquire the velocity v at N, hence $v^2=2ma$; and by sim. tri. $\dot{z} : \dot{x} :: 1\ (PR) : \frac{\dot{x}}{\dot{z}}$ that part of gravity which acts in the direction PO; but (Art. 202.) the forces of bodies revolving in circles, vary as the squares of the velocities divided by the radii; hence, $\frac{v^2}{a}$, or $2m : \frac{V^2}{r} :: 1 : \frac{\dot{x}}{\dot{z}}$, and $V^2 = \frac{2mr\dot{x}}{\dot{z}}$; but the square of the velocity at Q, or at P, is equal $4m \times BQ = 4m \times (n+a-y)$; but to the same radius, the centrifugal force is as the square of the velocity, and the centrifugal force at P corresponding to V, is $\frac{\dot{x}}{\dot{z}}$, it being equal to its centripetal force; hence, $\frac{2mr\dot{x}}{\dot{z}} : 4m \times (n+a-y) :: \frac{\dot{x}}{\dot{z}} : \frac{2}{r} \times (n+a-y)$ the centrifugal force at P in the curve; this therefore taken from $\frac{\dot{x}}{\dot{z}}$ the force of gravity in the direction PO, gives $\frac{\dot{x}}{\dot{z}} - \frac{2}{r} \times (n+a-y)$ the pressure at P. If the body begin to descend from E, and EB' be parallel to AW, and $B'W=a$, the pressure $= \frac{\dot{x}}{\dot{z}} - \frac{2}{r} \times (a-y)$.

Ex. 1. Let NA be the *quadrant of a circle* whose centre is W, then O coincides with W, and $r=a$; also, $\dot{z} : \dot{x} :: a : y$, and $\frac{\dot{x}}{\dot{z}} = \frac{y}{a}$; hence, the pressure $= \frac{y}{a} - \frac{2}{a} \times (n+a-y) = \frac{3y-2a-2n}{a}$, the weight of the body being unity. If the body begin its motion indefinitely near N, $n=0$, and the pressure $= \frac{3y-2a}{a}$.

If *NA* be convex to *AW*, the whole pressure at *P* = $\frac{\dot{x}}{\dot{z}} + \frac{2}{r} \times (n+a-y)$; or when $n=0$, it becomes $\frac{\dot{x}}{\dot{z}} + \frac{2}{r} \times (a-y)$.



Ex. Let *NA* be the *quadrant of a circle.* Then the pressure at $P = \frac{a-y}{a} + \frac{2}{a} \times (n+a-y)$. When the body comes to *A*, if $n=0$, the pressure $=3$, the weight of the body being unity.

Prop. CLXXX.

Let the body descend down the curve NA (Fig. last but one); *to find the point where it will quit the curve.*

The body quits the curve at the point where the pressure $=0$; or when $\frac{\dot{x}}{\dot{z}} - \frac{2}{r} \times (n+a-y) = 0$.

Ex. Let *NA* be the *quadrant of a circle.* Then $r=a$, and $\frac{\dot{x}}{\dot{z}} = \frac{y}{a}$; hence $\frac{y}{a} - \frac{2}{a} \times (n+a-y) = 0$. If $n=0$, $y=\frac{2}{3}a$. If n be not $= 0$, then $3y-2a-2n=0$, or $y=\frac{2}{3}(a+n)$. As y cannot be greater than a, the limit of n is $\frac{1}{2}a$; if n be greater than that, the body will leave the curve at *N*.

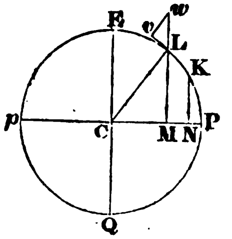
There is another rule given to solve this problem: that is, to make the fluxion of the velocity in the direction of the ordinate, a maximum. But to establish this principle, it must first be proved, that when a body descends along any curve, the fluxion of the velocity in the direction of the ordinate, cannot first increase and then decrease.

As the body quits the curve where the pressure $=0$, or when $\frac{\dot{x}}{\dot{z}} - \frac{2}{r} \times (n+a-y) = 0$; it is manifest, that when this quantity can never become $= 0$, the body will never quit the curve.

Prop. CLXXXI.

If the earth were a perfect sphere, how much would a plumb-line deviate from a perpendicular to the surface?

Let C be the centre of the earth, E the equator, P, p, the poles, draw LM perpendicular to PC, produce it to w, let Lw represent the centrifugal force at L, and draw wv perpendicular to the tangent vL. Let $CP = 1$, $CM = x$, $ML = y$, $c = \frac{1}{289}$ the centrifugal force at E, gravity being unity; then (Art. 202. Cor. 2.) $1 : y :: c : cy = Lw$, and $1 : x :: cy : Lv = cxy$ the force which draws the pendulum from the perpendicular. Now that the pendulum may rest, the force of gravity to bring it back must be $=$ this force; and that force (cxy) : gravity :: (s) the sine of deviation : 1 rad. hence $cxy = s$; but if $z = \sin. 2EL$, $xy = \frac{1}{2}z$; hence, $s = \frac{1}{2}cz$.



The deviation is greatest when z is greatest, or in lat. 45°.

PROP. CLXXXII.

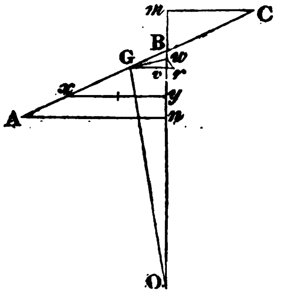
If a body be laid at any point K *on the earth's surface, supposed a sphere; to find the velocity with which it would arrive at* E.

Put $z = EL$, $b = KN$, $d = CN$, $v =$ velocity at L; then from last Prop. $cxy = Lv$ the accelerative force towards E; hence (Art. 82.) $v\dot{v} = -2mcxy\dot{z} =$ (as $-\dot{z} : \dot{y} :: 1 : x$) $2mcy\dot{y}$, and $\frac{v^2}{2} = mc(y^2 + C^2)$; but when $v = 0$, $y = b$; hence, $v =$ (when $y = 1$) $d\sqrt{2mc}$, the velocity at E.

PROP. CLXXXIII.

Let a uniform straight rod ABC *revolve about a vertical axis* BO, G *its center of gravity on the lower side* AB; *to find the velocity of the rod about* BO, *to preserve the axis vertical.*

Join OG, assume any point x, and draw An, Cm, Xy, Gv perpendicular to OB; put $AB = a$,



$An = b$, $Bn = \epsilon$, $Gv = e$, $BC = \alpha$, $Cm = \beta$, $Bm = \gamma$, $AC = p$, $vO = q$, $OB = d$, $m = 32\frac{1}{6}$ feet, $Br = x$, $v =$ velocity at the distance 1. Then $a : b ::$

$x : \frac{bx}{a} = xy$, $\overline{a : c} :: x : \frac{cx}{a} = By$, then $Oy = d - \frac{cx}{a}$; Hence, the velocity of the point $x = \frac{bvx}{a}$; and (Art. 202.) the centrifugal force of $x = \frac{bv^2x}{a}$; and the effect of this force on $\dot{x}$ to turn the axis about $O = \frac{bv^2}{a} \times x\dot{x} \times \left(d - \frac{cx}{a}\right) = \frac{bdv^2}{a} \times x\dot{x} - \frac{bcv^2}{a^2} \times x^2\dot{x}$, the fluent of which, when $x = a$, is $\frac{1}{2}abdv^2 - \frac{1}{3}abcv^2$ the efficacy of AB to turn BO about O. By the same process we get $\frac{1}{2}\alpha\beta dv^2 + \frac{1}{3}\alpha\beta\gamma v^2$ the efficacy of BC to turn BO about O in the opposite direction to that of AB. Now to find the force (f) acting parallel to the horizon to support the rod, draw Gw perpendicular to GO, Gvr to BO, and rw to Gw. Then as the tendency of G is to move in the direction wG, we have, $ac.f$: Gra. $:: vw : Gw$, and $ac.f. =$ Gra. $\times \frac{vw}{wG} =$ Gra. $\times \frac{Gv}{GO}$; but a force (f) acting parallel to the horizon to support the rod : $ac.f.$ $\left(= \text{Gra.} \times \frac{Gv}{GO}\right) :: Gr : Gw :: GO : vO$, therefore $f =$ Gra. $\times \frac{Gv}{vO} = \frac{m \times AC \times Gv}{vO} = \frac{mpe}{q}$. Hence, $\frac{1}{2}abdv^2 - \frac{1}{3}abcv^2 + \frac{mpe}{q} = \frac{1}{2}\alpha\beta dv^2 + \frac{1}{3}\alpha\beta\gamma v^2$, and $v = \sqrt{\frac{mpe}{\frac{1}{2}dq \times (\alpha\beta - ab) + \frac{1}{3}q \times (abc + \alpha\beta\gamma)}}$; in all cases therefore where v is possible, the axis will remain at rest.

If the efficacy on the right of BO be greater than that on the left, the rod will ascend.

Instead of a line we might have supposed a plane body, and the calculation would have gone on, on the same ground.

PROP. CLXXXIV.

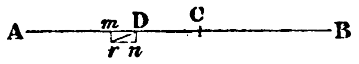
A candle 12 *inches long burns down* 1 *inch the first hour; and it burns down with a velocity which is as the distance of two inches from the top at first; in what time will the whole candle be burnt out?*

Put x = any distance burnt down, v = corresponding velocity of burning down, t'' = time, and let the velocity be represented by $m \times (2+x)$; then $\dot{t} = \frac{\dot{x}}{m \times (2+x)}$, and $t = \frac{1}{m} \times$ h. l. $(2+x)$; but when $x = 0$, $t = 0$, and $t = \frac{1}{m} \times$ h. l. $\frac{2+x}{2}$; now when $t = 3600''$, $x = 1$ inch; hence, $3600 = \frac{1}{m} \times$ h. l. $\frac{3}{2}$, and $\frac{1}{m} = \frac{3600}{\text{h. l. } \frac{3}{2}}$; hence, $t = \frac{3600}{\text{h. l. } \frac{3}{2}} \times$ h. l. $\frac{2+x}{2}$, and when $x = 12$, $t = \frac{3600}{\text{h. l. } \frac{3}{2}} \times$ h. l. $7 = 17277'' = 4^{h}.\ 47'.\ 57''$.

PROP. CLXXXV.

If a small cylindrical rod AB *pass freely through a hole at* C, *about which point it has a rotatory motion, and the rod be unequally divided at* C; *to find the time in which the rod will quit the hole.*

Bisect the rod in D, put $AD = DB = a$, $CD = x$ after any time t, d the first value of x, v = velocity of D in



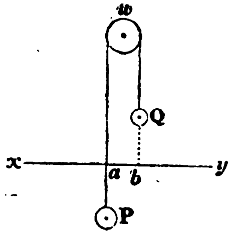
the direction of the rod at the time t, and at the distance 1 let c = velocity about C, and y = distance of any point of the rod from C; then $1 : y :: c : cy$ the velocity about C at the distance y; and (Art. 202.) $c^2 y =$

centrifugal force at that distance, the force of gravity being $32\frac{1}{6}$ feet; hence, $cy^2\dot{y}$ is the fluxion of the force, and the fluent is $\frac{1}{2}c^2y^2$; and making $y = CA$, CB, or $a+x$, $a-x$, the centrifugal forces of CA, CB, are $\frac{1}{2}c^2\times(a+x)^2$, $\frac{1}{2}c^2\times(a-x)^2$, and their difference is $2c^2ax$ the motive force by which the rod is urged in the direction CA, and this divided by $2a$ the quantity of matter in the rod, gives c^2x the accelerative force in the same direction. Hence (Art. 82. Cor.) $v\dot{v} = c^2x\dot{x}$, and the correct fluent is $v = c\sqrt{x^2-d^2}$; and $\dot{t} = \frac{\dot{x}}{c\sqrt{x^2-d^2}}$; hence (Art. 45. Ex. 4.) the correct fluent is (when $x=a$) $t = \frac{1}{c\sqrt{2}} \times$ h. l. $\frac{a+\sqrt{a^2-d^2}}{a}$ the time required.

PROP. CLXXXVI.

Let two weights P, Q, *be connected by a string passing over a pulley (supposed without weight), and begin their motions from* a, b, *in the same horizontal line* xy, *and at the same time to begin to vibrate; to find the number of vibrations whilst they describe any given space.*

Put half the length of the string $= a$, $x = aP = bQ$, then $a \pm x$ = the lengths of the two pendulums, the upper



sign belonging to P and the lower to Q, $s = 16\frac{1}{12}$ feet $=$ 193 inches, $r = 39{,}2$ inches, $m = \frac{P-Q}{P+Q}$ the accelerating force; then ms = space described in the first second; hence, $ms : x :: 1''^2 : t''^2 = \frac{x}{ms}$, and $t = \frac{x^{\frac{1}{2}}}{\sqrt{ms}}$ the time of describing x; hence, $\dot{t} = \frac{x^{-\frac{1}{2}}\dot{x}}{2\sqrt{ms}}$. Now $\sqrt{r} : \sqrt{a \pm x} :: 1'' : \frac{\sqrt{a \pm x}}{\sqrt{r}}$ the times of vibrations of P and Q respectively; hence, $\frac{\sqrt{a \pm x}}{\sqrt{r}} : \frac{x^{-\frac{1}{2}}\dot{x}}{2\sqrt{ms}}$ :: 1 oscillation : $\frac{\sqrt{r}}{2\sqrt{ms}} \times \frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{a \pm x}}$ the fluxion of an oscillation corresponding to $\dot{t}$, or $\dot{x}$ of descent or ascent.

Now for P, put $\sqrt{a+x} = y$, then $\frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{a+x}} = \frac{2\dot{y}}{\sqrt{y^2 - a}}$, and the fluent of $\frac{\sqrt{r}}{2\sqrt{ms}} \times \frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{a+x}}$ is $\frac{\sqrt{r}}{2\sqrt{ms}} \times [2 \text{ h. l. } (y + \sqrt{y^2 - a})] = \frac{\sqrt{r}}{2\sqrt{ms}} \times [2 \text{ h. l. } (\sqrt{a+x} + \sqrt{x}) + C\}$ the number of vibrations whilst P descends down x; but when $x = 0$, the number of oscillations $= 0$; and the correct fluent when $x = a$ is $\frac{\sqrt{r}}{\sqrt{ms}} \times \text{h. l. } (\sqrt{2} + 1)$ which is always the same whatever the length of the string may be.

For Q, put $\sqrt{a - x} = y$, then $\frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{a - x}} = \frac{-2}{\sqrt{a}} \times \frac{\sqrt{a} \times \dot{y}}{\sqrt{a - y^2}}$, and the fluent of $\frac{\sqrt{r}}{2\sqrt{ms}} \times \frac{-2}{\sqrt{a}} \times \frac{\sqrt{a} \times \dot{y}}{\sqrt{a - y^2}}$

is $\frac{-\sqrt{r}}{\sqrt{ms}} \times$ (cir. arc (A) whose rad. $= \sqrt{a}$, sin. $= y$ or $\sqrt{a-x}+C$) and the correct fluent is $\frac{-\sqrt{r}}{\sqrt{ms}} \times \frac{1}{\sqrt{a}} \times (A-q)$ q being a quadrantal arc; and when $x=a$, the number of vibrations through the whole of Q's ascent $= \frac{\sqrt{r}}{\sqrt{ms}} \times \frac{1}{\sqrt{a}} \times q =$ (as $p : q :: 1 : \sqrt{a}$, putting p for the quadrantal arc whose radius $=1$) $\frac{\sqrt{r}}{\sqrt{ms}} \times p$, which is therefore the same, whatever be the length of the string.

Prop. CLXXXVII.

If a pendulum vibrating seconds at the earth's surface be carried uniformly upwards at the rate of a radius of the earth in 24 hours; how much will the pendulum lose?

Put r = radius of the earth, x = any distance of the pendulum from the center, $b = 86400''$ in 24 hours, t = time of ascent to the distance x; then (as the time of vibration varies inversely as the square root of the force) $\sqrt{r^2} : \sqrt{x^2}$, or $r : x :: 1'' : \frac{x}{r}$ the time of a vibration at the distance x; also $r : x-r :: b : t = \frac{b}{r} \times (x-r)$, and $\dot{t} = \frac{b}{r}\dot{x}$; hence, $\frac{x}{r} : \frac{b}{r} \times \dot{x} :: 1$ oscillation : $b \times \frac{\dot{x}}{x}$ the fluxion of an oscillation during $\dot{x}$ of ascent, whose fluent is $b \times$ h. l. $x + C$; but when $x = r$, the number of oscillations = 0; hence, the correct fluent is $b \times$ h. l. $\frac{x}{r}$ the number of oscillations in ascending $x - r$. But the number of seconds in ascending $x - r = \frac{b}{r} \times (x - r)$ = number of oscillations at the earth's surface; hence, the time lost $= \frac{b}{r} \times (x - r) - b \times$ h. l. $\frac{x}{r}$; and when $x = 2r$, the time lost $= b \times (1 -$ h. l. $2) = 86400 \times 0{,}30685286 = 7^h.\ 21'.\ 52''$.

PROP. CLXXXVIII.

If the same pendulum be carried uniformly down to the center of the earth, to find the time lost.

The same notation remaining, we have (as gravity varies within the earth as the distance from the center) $\frac{1}{\sqrt{r}} : \frac{1}{\sqrt{x}} :: 1'' : \frac{r^{\frac{1}{2}}}{x^{\frac{1}{2}}}$ the time of one vibration at the distance x from the center; also $r : r - x :: b : t = \frac{b}{r} \times (r - x)$, and $\dot{t} = -\frac{b}{r} \times \dot{x}$; hence, $\frac{r^{\frac{1}{2}}}{x^{\frac{1}{2}}} : -\frac{b}{r} \times \dot{x} ::$ 1 oscillation : $\frac{b}{r^{\frac{3}{2}}} \times -x^{\frac{1}{2}}\dot{x}$ the fluxion of an oscillation during the descent $\dot{x}$, whose correct fluent is $\frac{2b}{3r^{\frac{3}{2}}} \times (r^{\frac{3}{2}} - x^{\frac{3}{2}})$ the number of oscillations in descending $r - x$; and when $x = 0$, the number $= \frac{2}{3}b$; hence the time lost $= b - \frac{2}{3}b = \frac{1}{3}b = 8$ hours.

Prop. CLXXXIX.

A pendulum 39,2 *inches long, increased uniformly one-tenth of an inch in* 24 *hours; how much did the clock lose that day?*

Put $a = 39{,}2$, $m = 0{,}1$ inch, $b = 86400''$, $x =$ the increase of the length in the time t''; then $m : \dot{x} :: b : \dot{t} = \frac{b}{m}\dot{x}$; and $\sqrt{a} : \sqrt{a+x} :: 1'' : \frac{\sqrt{a+x}}{\sqrt{a}}$ the time of one vibration when the length is $a+x$; hence, $\frac{\sqrt{a+x}}{\sqrt{a}} : \frac{b}{m}\dot{x} ::$ 1 oscillation : $\frac{b\sqrt{a}}{m} \times \frac{\dot{x}}{\sqrt{a+x}}$ the fluxion of an oscillation corresponding to the increase $\dot{x}$ of the pendulum, the fluent of which is $\frac{2b\sqrt{a}}{m} \times (\sqrt{a+x} + C)$; but when $x = 0$, the number of vibrations $= 0$, and the correct fluent is $\frac{2b\sqrt{a}}{m} \times (\sqrt{a+x} - \sqrt{a}) =$ (when $x = m$) $\frac{2b\sqrt{a}}{m} \times \sqrt{a+m} + \sqrt{a}$ the number of vibrations performed in one day. Hence the clock lost $b - \frac{2b\sqrt{a}}{m} \times (\sqrt{a+m} - \sqrt{a})$ seconds. But $\sqrt{a+m} - \sqrt{a} = \frac{m}{2a^{\frac{1}{2}}} - \frac{m^2}{8a^{\frac{3}{2}}}$, neglecting the other terms on account of their smallness. Hence, the seconds lost in 24 hours is $\frac{bm}{4a} = 55'', 1$.

As the loss of time is in preparation to m, for any other increase w, we have $m : w :: 55'', 1 : 55'', 1 \times \frac{w}{m}$ the loss of time.

In general, if the vibration of a pendulum continually vary, either from the variation of force or length, the computation is made on the same principle.

Prop. CXC.

To compare the force of gravity with the force (f) *by which the surface of a fluid descends in a vessel emptying itself through a small hole at the bottom.*

Put z = area of the surface, n = area of the orifice, x = depth of the fluid, t = time of emptying the depth x, $m = 16\frac{1}{12}$ feet; then (Art. 193.) $\frac{n}{z} \times \sqrt{mx}$ is the velocity (v) with which the surface descends. Now if $2m$ represent the force of gravity, $v\dot{v} = f\dot{x}$; but $\dot{v} = n\sqrt{m} \times \left(\frac{\dot{x}}{2zx^{\frac{1}{2}}} - \frac{x^{\frac{1}{2}}\dot{z}}{z^2}\right)$ and $f = \frac{v\dot{v}}{\dot{x}} = n^2m\left(\frac{1}{2z^2} - \frac{x\dot{z}}{z^3\dot{x}}\right)$.

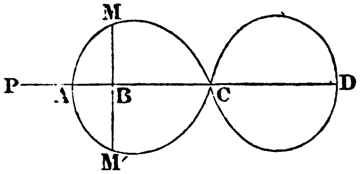
For a cylinder or prism standing on its base, z is constant, and $\dot{z} = 0$; hence, $f = \frac{n^2m}{2z^2}$ a constant retarding force.

If $\frac{1}{2z^2} - \frac{x\dot{z}}{z^3\dot{x}} = 0$, then $f = 0$, and the surface descends uniformly; hence $\frac{\dot{x}}{x} = \frac{2\dot{z}}{z}$, and h. l. x = 2 h. l. z, and $x = z^2$; but if y be the ordinate, corresponding to the abscissa x, z varies as y^2, hence x varies as y^4 the equation of the curve.

Prop. CXCI.

No oval figure can be squared, unless it cuts itself.

Let $PB = x$, $BM = y$; assume $\overline{x^2 - px + q}|^{\frac{3}{2}}$, whose fluxion is $\frac{3}{2}\sqrt{x^2 - px + q} \times (2x\dot{x} - p\dot{x})$; let $y = \sqrt{x^2 - pr + q} \times (2x - p)$; then the square root of a quantity being + or −, the ordinate y has two equal values BM; BM' lying on the opposite sides of the abscissa. Now $y\dot{x} = \sqrt{x^2 - px + q} \times (2x\dot{x} - p\dot{x})$; and the quantity



without the radical sign being the fluxion of the quantity under it, the fluent (Art. 40.) is $\frac{2}{3} \times \overline{x^2 - px + q}|^{\frac{3}{2}}$, and under no other circumstances can the fluent be found in finite terms; this appears from Art. 39 *. Now make $x^2 - px + q = 0$, and let the roots $\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$, be m, n; take $PA = m$, $PD = n$, and then, as in each case $y = 0$, the curve cuts the axis at A, D; but $2x - p = 0$ is the limiting equation to $x^2 - px + q = 0$, and therefore its root $\frac{p}{2}$ lies between m and n; let $PC = \frac{p}{2}$, and then as $y = 0$, the curve cuts the axis at C between A and D; also $AC = CD$, PC being an arithmetic mean between PA, PD. For x put $\frac{p}{2} \pm d$, that is, take an ordinate

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* If P be any fluent, it's fluxion must come from it as a basis; that is, the fluxional part, considered by itself, must depend on and be derived from P. If therefore $P = \overline{x^2 - px + q}|^{\frac{3}{2}}$, the fluxional part must depend on $x^2 - px + q$, and $\frac{3}{2}$. Now we know by the above article, that the fluxional part must be $2x\dot{x} - p\dot{x}$, or some constant multiple of it, to preserve the same basis, or that the fluent may be found. If therefore the fluxional part were not of that form, two different fluxions might have the same basis, or the same fluent might have two different fluxions.

in each part equidistant from C, and put this for x into the value of y, and we get $y = \sqrt{q - \frac{p^2}{4} + d^2} \times \pm 2d$; hence the two ordinates equidistant from C, are equal; and the two parts of the figure are similar and equal.

To find the area of the two ovals, the fluent of $y\dot{x} = \frac{2}{3} \times \overline{x^2 - px + q}|^{\frac{3}{2}}$, make $x = \frac{p}{2} - \sqrt{\frac{p^2}{4} - q}$, and $\frac{p}{2}$, or PA, PC, and the area $AMCBA$ lies between those values of x; substitute therefore $\frac{p}{2} - \sqrt{\frac{p^2}{4} - q}$ and $\frac{p}{2}$ in the fluent and take the difference and we get $\frac{2}{3} \times \overline{q - \frac{p^2}{4}}|^{\frac{3}{2}} = AMCBA$; hence, $\frac{8}{3} \times \overline{q - \frac{p^2}{4}}|^{\frac{3}{2}} =$ the whole area.

Cor. Hence, a circle cannot be squared.

LEMMA.

If N consist of functions of x and y, and you take it's fluxion, and the fluxion of that fluxion, first supposing x constant and then y; secondly, supposing y constant and then x; the two functions, omitting the fluxional factors, will be the same.

For the fluxions of the functions of x and y being taken separately, it makes no difference which you take first.

Prop. CXCII.

Let $P\dot{x} + Q\dot{y}$ *be a fluxion in which* P *and* Q *are functions of* x *and* y; *to find when the fluent can be found.*

Let the fluxion of P be $p\dot{x} + q\dot{y}$, and that of Q be $r\dot{x} + s\dot{y}$; then if $r = q$ the fluent can be found. For if

N be the fluent, then Q arises from taking the fluxion of N and then again taking it's fluxion, first supposing x constant and then y; and r arises from first supposing y constant and then x; hence by the Lemma, if the fluent can be expressed by any quantity N, r must $= q$.

Prop. CXCIII.

Given two connected variable quantities x, y, *in a fluxion* ($\dot{\mathrm{F}}$); *to find when* F *can be found.*

Ex. 1. Let $\dot{F} = y^3\dot{y}\sqrt{a^8+x^8} + \frac{y^4x^7\dot{x}}{\sqrt{a^8+x^8}}$; then if x be constant, the fluent is $\frac{1}{4}y^4\sqrt{a^8+x^8}$, and if y be constant, the fluent is $\frac{1}{4}y^4\sqrt{a^8+x^8}$, the same as before; hence $F = \frac{1}{4}y^4\sqrt{a^8+x^8}$.

The reason is this; when you take the fluxion of F, you first make one of the quantities variable, and then the other; therefore the fluent of each separately must be the same, if it can be found.

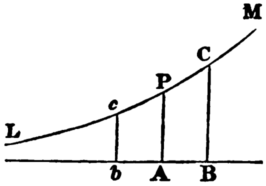
Ex. 2. Let the quantity be $9x^2y^3\dot{x}+20x^3y^3\dot{y}$. If x only be variable, the fluent is $3x^3y^3$; if y be only variable, the fluent is $5x^3y^4$; and these quantities not being equal, the fluent cannot be found.

Prop. CXCIV.

Let $\mathrm{y}=\mathrm{a}+\mathrm{bz}+\mathrm{cz}^2+\mathrm{dz}^3+$ *&c. and when* $\mathrm{z}=0$, *let* v *be the value of* y, *and* $\dot{\mathrm{v}}$, $\ddot{\mathrm{v}}$, $\dddot{\mathrm{v}}$ *&c. the values of* $\dot{\mathrm{y}}$, $\ddot{\mathrm{y}}$, $\dddot{\mathrm{y}}$, *&c.* z *flowing uniformly; then* $\mathrm{y} = \mathrm{v} + \frac{\mathrm{z}\dot{\mathrm{v}}}{\dot{\mathrm{z}}} + \frac{\mathrm{z}^2\ddot{\mathrm{v}}}{1.2\dot{\mathrm{z}}^2} + \frac{\mathrm{z}^3\dddot{\mathrm{v}}}{1.2.3\dot{\mathrm{z}}^3} + \frac{\mathrm{z}^4\ddddot{\mathrm{v}}}{1.2.3.4\dot{\mathrm{z}}^4} +$ *&c. where the law of continuation is manifest.*

As $y=a+bz+cz^2+dz^3+$&c. if $z=0$, $a=y=v$ by supposition; take the fluxion and $\frac{\dot{y}}{\dot{z}}=b+2cz+3dz^2$ +&c. and when $z=0$, $b=\frac{\dot{y}}{\dot{z}}=\frac{\dot{v}}{\dot{z}}$; take the fluxion of the last equation, and $\frac{\ddot{y}}{\dot{z}^2}=2c+2.3dz+$&c. and when $z=0$, $2c=\frac{\ddot{y}}{\dot{z}^2}$ and $c=\frac{\ddot{v}}{2\dot{z}^2}$; take the fluxion of the last equation, and $\frac{\dddot{y}}{\dot{z}^3}=2.3.d+$&c. and $d=\frac{\dddot{y}}{2.3\dot{z}^3}=\frac{\dddot{v}}{2.3\dot{z}^3}$, &c. &c.; hence, $y=v+\frac{z\dot{v}}{1\dot{z}}+\frac{z^2\ddot{v}}{1.2\dot{z}^2}+\frac{z^3\dddot{v}}{1.2.3\dot{z}^3}+$ &c.

Cor. 1. Let LM be a curve, the ordinate $AP=v$, $BC=y$, $AB=z$, also let $AB=\dot{z}$; then $BC=y=v+$



$\dot{v}+\frac{\ddot{v}}{1.2}+\frac{\dddot{v}}{1.2.3}+$&c. and $y-v=\dot{v}+\frac{\ddot{v}}{1.2}+\frac{\dddot{v}}{1.2.3}$ + &c. Now $y-v$ is the increment of y; hence, this equation shows how the fluxion of y of each order contributes to produce the increment of y.

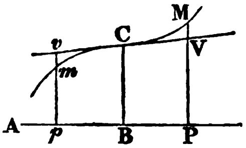
Cor. 2. If the ordinate bc be taken on the other side of AP, so that z may be negative, then bc, or y, $=$ $a-bz+cz^2-dz^3+$&c. and $bc=y=v-\frac{z\dot{v}}{1\dot{z}}+\frac{z^2\ddot{v}}{1.2\dot{z}^2}$

$-\frac{z^3\dddot{v}}{1\,.\,2\,.\,3\dot{z}^3}$ + &c. hence, $BC + bc = 2v + \frac{2z^2\ddot{v}}{1\,.\,2\dot{z}^2} + \frac{2z^4\ddddot{v}}{1\,.\,2\,.\,3\,.\,4\dot{z}^4}$ + &c.

COR. 3. Let $\dot{v} = 0$; then if $\ddot{v}$ be *positive* the ordinate is a *minimum*; if *negative* a *maximum*; that is, $\dot{y}$ is *less* in the *former* and *greater* in the *latter* case, than in the adjoining parts. For if $\dot{v} = 0$, $BC = v + \frac{z^2\ddot{v}}{1\,.\,2\dot{z}^2}$ + &c. and $bc = v + \frac{z^2\ddot{v}}{1\,.\,2\dot{z}^2}$ − &c. therefore if $\ddot{v}$ be positive, and z be taken very small, all the other terms may be neglected, and BC, bc, are each greater than AP; but if $\ddot{v}$ be negative, then BC, bc, are less than AP. But if $\dot{v} = 0$, $\ddot{v} = 0$, and $\dddot{v}$ does not vanish, so that $BC = v + \frac{z^3\dddot{v}}{1\,.\,2\,.\,3\dot{z}^3}$ + &c. $bc = v - \frac{z^3\dddot{v}}{1\,.\,2\,.\,3\dot{z}^3}$ + &c. In this case, BC is *greater* than AP, bo is *less*, so that AP is neither a maximum or minimum. In general, if the first and subsequent fluxions vanish, the ordinate is (for the same reason) a *minimum* or *maximum* when the number of fluxions which vanish is 1, 3, 5, &c. or any *odd* number; it is a *minimum* when the fluxion *next* to those which vanish, is *positive*; but a *maximum* when *negative*. But if the number of fluxions which vanish be an *even* number, the ordinate is then neither a *maximum* nor minimum.

COR. 4. The points of contrary flexure are generally determined by making $\ddot{y} = 0$, or infinity, $\dot{x}$ being constant (Art. 79.); but this does not always happen. Let C be the point of contrary flexure, and PM, pm, meet the tangent at C in V, v; then (Art. 23.) $PV = v + \frac{z\dot{v}}{\dot{z}}$, $pv = v - \frac{z\dot{v}}{\dot{z}}$; and when $\ddot{v} = 0$, $PM = v + \frac{z\dot{v}}{\dot{z}} +$

$\frac{z^3\dddot{v}}{1.2.3\dot{z}^3}$ + &c. $pm = v - \frac{z\dot{v}}{\dot{z}} - \frac{z^3\dddot{v}}{1.2.3\dot{z}^3}$ + &c. hence, if $\dddot{v}$ be positive, and *BP*, *bp*, be taken so small that all the other terms may be neglected, *PM* will be greater than *PV*, and *pm* less than *pv*; and whether $\dddot{v}$ be positive or negative, the arcs *CM*, *Cm*, lie on different



sides of *vCV*, and therefore *C* is a point of contrary flexure. As $\dot{v}$ has regard to the tangent, it is the other fluxions $\ddot{v}$, $\dddot{v}$, $\ddddot{v}$, &c. which denote whether the curve lies above or below the tangent. If $\ddot{v}$, $\dddot{v}$, and $\ddddot{v}$ have a real value, *PM*, *pm* will be both greater or less than *PV*, *po*, and hence *C* will be no point of contrary flexure. Or in general, if $\ddot{v}$, $\dddot{v}$, $\ddddot{v}$, &c. vanish, the number of these points being *odd*, and the fluxion of the next order has a real value, then *C* is a point of contrary flexure; but if the number be *even*, *C* is not such a point.

Ex. Let $y = a^4 + x^4$; then $\dot{y} = 4x^3\dot{x}$, and ($\dot{x}$ being constant) $\ddot{y} = 12x^2\dot{x}^2$, $\dddot{y} = 24x\dot{x}^3$, $\ddddot{y} = 24\dot{x}^4$. Now if $\ddot{y} = 0$, it must be that $x = 0$, because $\dot{x}$ is constant; on that account $\dot{y} = 0$, $\dddot{y} = 0$; but $\ddddot{y}$ is a real quantity; hence, *C* is not a point of contrary flexure; but it is a maximum; for as $\dot{y} = 0$, the tangent is parallel to the base, and the curve on each side lies above the tangent; for $PM = v + \frac{z^4\ddddot{v}}{1.2.3.4\dot{z}^4}$, and *pm* is the same quantity; hence, *M*, *m*, lie above the tangent.

The further uses of this proposition, are as follows.

Let $z = \log. y$; then (Art. 45.) $\dot{y} = \frac{y\dot{z}}{M}$; hence

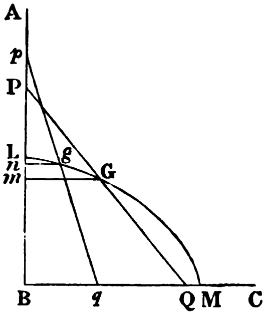
($\dot{z}$ constant), $\ddot{y} = \frac{\dot{y}\dot{z}}{M} = \frac{y\dot{z}^2}{M^2}$; $\dddot{y} = \frac{\dot{y}\dot{z}^2}{M^2} = \frac{y\dot{z}^3}{M^3}$, &c. &c. but when $z=0$, $y=1=v$; hence, $\dot{v}=\dot{y}=(y=1)\frac{\dot{z}}{M}$; $\ddot{v}=\ddot{y}=\frac{\dot{z}^2}{M^2}$; $\dddot{v}=\dddot{y}=\frac{\dot{z}^3}{M^3}$, &c. &c. therefore $y=1+\frac{z}{M}+\frac{z^2}{2M^2}=\frac{z^3}{2.3M^3}+$ &c.

Again, let z = arc of circle, rad. $=a$, cos. $=y$; then, (Art. 46.) $\frac{\dot{y}}{\dot{z}}=\frac{\sqrt{a^2-y^2}}{a}$, $\frac{\dot{y}^2}{\dot{z}^2}=\frac{a^2-y^2}{a^2}$; hence, $\frac{2\dot{y}\ddot{y}}{\dot{z}^2}=\frac{-2y\dot{y}}{a^2}$, and $\frac{\ddot{y}}{\dot{z}^2}=-\frac{y}{a^2}$; therefore $\frac{\dddot{y}}{\dot{z}^3}=-\frac{\dot{y}}{a^2}$, and $\frac{\ddddot{y}}{\dot{z}^2}=-\frac{\ddot{y}}{a^2}=\frac{y\dot{z}^2}{a4}$, &c. &c. but when $z=0$, $y=a=v$; substitute therefore a for y in the above values, and $\dot{v}$ for $\dot{y}$, and we get $\frac{\dot{v}}{\dot{z}}=0$; $\frac{\ddot{v}}{\dot{z}^2}=-\frac{1}{a}$; $\frac{\dddot{v}}{\dot{z}^3}=-\frac{\dot{y}}{\dot{z}a^2}=0$; $\frac{\ddddot{v}}{\dot{z}^4}=\frac{y}{a^4}=\frac{1}{a^3}$; &c. &c. hence, $y=a-\frac{z^2}{2a}+\frac{z^4}{2.3.4a^3}$ − &c.

Prop. CXCV.

Let BC *be an horizontal plane,* AB *a plane perpendicular to it,* PQ *a slender cylindrical rod, and let the planes and ends of the rod be perfectly smooth, and the rod to be first in the position* pq, *and left to slide*; *to find when the end* P *will quit the plane.*

Let *G* be the center of gravity of the rod *PQ*; then whilst the end *P* keeps in the plane *AB*, *G* will describe a circular arc *g G*, *g* being the center of gravity of the rod *pq* in it's first position (Lem. page 319.). Draw *Gm*, *gn* perpendicular to *AB*. Now the planes and ends of the rod being perfectly smooth, there is nothing to hinder the free descent of the center of gravity in that arc, all the effect of the planes is to make



the center describe the curve; G therefore descends down that curve as a heavy body would; hence (Prop. 180.) if $mB = \frac{2}{3} nB$, G is the point of the center of gravity when P leaves the plane; for if it continued in the plane it would describe the circular arc.

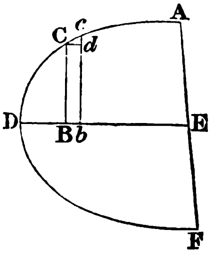
If the center of gravity be not in the middle of the rod, G describes an ellipse, and the point where the rod quits the plane is determined by the same Prop.

Prop. CXCVI.

To find the curve which a perfectly flexible line ADF *void of gravity, would form itself into by the force of the wind acting against it.*

Let DE be the axis, perpendicular to which draw BC, bc, indefinitely near, and Cd parallel to Bb. Put $DB = x$, $BC = y$, $DC = z$, $Cd = \dot{x}$, $dc = \dot{y}$, $Cc = \dot{z}$. Then the quantity of fluid acting on Cc is as cd or $\dot{y}$, and its effect perpendicular to the curve, as $\frac{\dot{y}}{\dot{z}}$; hence, the whole effect on $\dot{z}$ perpendicular to the curve, is as $\frac{\dot{y}^2}{\dot{z}}$; let $\dot{z}$ be constant, then the radius of curvature at

$C = \frac{-\dot{x}\dot{z}}{\ddot{y}}$; hence (Pr. 173.) $\frac{\dot{y}^2}{\dot{z}} \propto \frac{-\ddot{y}}{\dot{x}\dot{z}}$; or as $\dot{z}$ is constant, assume a a constant quantity so that $\frac{\dot{y}^2}{\dot{z}} = \frac{-a\dot{z}\ddot{y}}{\dot{x}\dot{z}}$



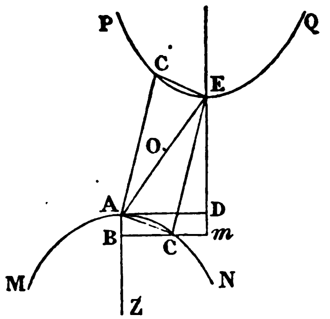
$= \frac{-a\ddot{y}}{\dot{x}}$, and $\frac{\dot{x}}{\dot{z}} = \frac{-a\ddot{y}}{\dot{y}^2}$, whose fluent is $\frac{x}{\dot{z}} = \frac{a}{\dot{y}} + C$; but at D, $\dot{y} = \dot{z}$, and $x = 0$; therefore the correct fluent is $\frac{x}{\dot{z}} = \frac{a}{\dot{y}} - \frac{a}{\dot{z}}$, and $a\dot{z} = (a+x) \times \dot{y}$ the equation to the catenary.

Prop. CXCVII.

Let A, E, *be two given points, to find the locus of all the points* C, *to which a body may fall from* A, E, *in the same time.*

Draw AD, BCm, parallel, and ED, AZ perpendicular to the horizon, and produce BC to meet ED in m. Put $AD=a$, $DE=b$, $AB=x$, $BC=y$, then $AC^2 = x^2+y^2$, $Cm = a \sim x$, $Em = bty$, $EC^2 = \overline{a \sim x}\rvert^2 + (b+y)^2$; and the squares of the times of descents being as the squares of the lengths divided by the heights, $\frac{x^2+y^2}{x} = \frac{\overline{a \sim x}\rvert^2 + \overline{b+y}\rvert^2}{b+x}$; hence, $by^2 - bx^2 + 2axy - (a^2+b^2) \times x = 0$, an equation to an hyperbola passing through A, for when $x=0$, $y=0$. Take the fluxion of this equation, and we get $\dot{x} : \dot{y} :: 2by + 2ax : 2bx -$

$2ay+a^2+b^2$, and making x, y, evanescent, it appears that at A, $\dot{x}$ is indefinitely less than $\dot{y}$; hence, DA is a tangent to the curve at A.



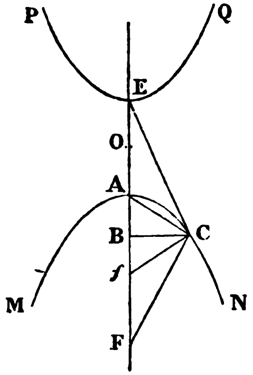
If the parallelogram $ACEC'$ be completed, the point C' will describe a like curve $PC'EQ$ as C does, and will therefore be the opposite hyperbola, and has this property, that the times of descent from C' to A and E are equal.

As A, E, are opposite points in the opposite hyperbolas; a tangent to E is parallel to the horizon; and if A, E, be bisected in O, that point is the center of the hyperbola.

If $a=0$, the equation becomes $y^2=x^2+bx$ an equation to an equilateral hyperbola, of which, b, or AE, is the axis. Hence, when EA is perpendicular to the horizon, the times of descent from the extremities of the axis to any point C in the curve, are equal; thus exhibiting some analogy between the equilateral hyperbola and the circle.

Draw CF perpendicular to AC, and Cf to EC; then as the times down AC, EC are equal, $\frac{EC}{\sqrt{EB}}=\frac{AC}{\sqrt{AB}}=\sqrt{AF}$; hence, $EB : EC :: EC : AF$. Put $a=AO$, $x=AB$, then by the property of the hyperbola, $2ax+x^2=BC^2$, therefore $2ax+2x^2=BC^2+x^2=AC^2=x+AF=x\times(x+BF)$; hence $2a+x=BF$, or $BE=$

BF; ECF is therefore an isosceles triangle. Also,



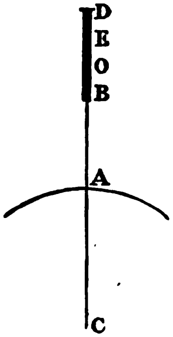
$\frac{AC}{\sqrt{AB}} = \frac{EC}{\sqrt{EB}} = \sqrt{Ef}$, and $AC^2 = AB \times Ef$, that is $2ax + 2x^2 = x \times Ef = x \times (EB + Bf) = x \times (2a + x + Bf)$; hence, $Bf = x = BA$; therefore ACf is an isosceles triangle. As $Bf = BA$, and $BF = BE$, therefore $Ff = AE$ a constant quantity. Hence, the times through Cf, CF are equal, CF being $= CE$, and equally inclined to the horizon; also $Cf = AC$. Such are the properties of the equilateral hyperbola.

In like manner it appears, that if C be a fixed point, the loci of all the points in which the times of descent from E to A and from A to C, are the same, is in general a curve of *three* dimensions; and when C is vertical to E, the locus of C is a circle, and a straight line bisecting EC and perpendicular to it.

Prop. CXCVIII.

Let C *be the center of the Earth,* A *it's surface,* DB *a slender, uniform rod, lying in the direction of the radius* CA; *to find the point* O *of the rod into which the whole quantity of matter in the rod must be collected, that the attraction of the rod to the Earth may remain the same. This point* O *we will call the center of attraction.*

Let $CA = r$, gravity at $A = 1$, $CD = c$, $CB = b$, $DB = a$ representing the quantity of matter in the rod,



$CE = x$; then the fluxion of the attraction $= \frac{r^2\dot{x}}{x^2}$; whose fluent is $-\frac{r^2}{x} + C$; but when $x = CB$ the attraction $= 0$; hence, the whole attraction $= r^2 \times \left(\frac{1}{b} - \frac{1}{c}\right) = \frac{r^2 \times (c - b)}{bc}$ $= \frac{r^2 a}{bc}$. Let $z = CO$; then the attraction of a at $O =$ $\frac{r^2 a}{z^2}$; hence, $\frac{r^2 a}{bc} = \frac{r^2 a}{z^2}$, and $z = \sqrt{bc}$. Hence, at different distances of the rod from the earth, O is not a fixed point; and in general, it will not be so.

Prop. CXCIX.

Given as in the last proposition, to find the velocity of the rod.

When we use this theorem $v\dot{v} = -2mF\dot{x}$ the space and corresponding velocity are those of the point where the whole force acts. Now let $c = CD$ the first value of CD, $x = CD$ any other value; then as the force is applied at O, v must correspond to the same point, and $\dot{x}$ must represent the value of $\dot{CO}$. Now $CO = \overline{x^2 - ax}|^{\frac{1}{2}}$,

$\dot{C}O = \dfrac{2x\dot{x} - a\dot{x}}{2 \times \overline{x^2 - ax}|^{\frac{1}{2}}}$; also, $F = \dfrac{r^2}{x^2 - ax}$; hence, $v\dot{v} = -mr^2 \times \overline{x^2 - ax}|^{-\frac{3}{2}} \times (2x\dot{x} - a\dot{x})$, and the correct fluent is $v^2 = 4mr^2 \times (\overline{x^2 - ax}|^{-\frac{1}{2}} - \overline{c^2 - ac}|^{-\frac{1}{2}})$; hence, $v = \sqrt{4mr} \times \sqrt{\overline{x^2 - ax}|^{-\frac{1}{2}} - \overline{c^2 - ca}|^{-\frac{1}{2}}}$. If c be infinite, and $a = 0$, so that the body becomes a point, then at the earth's surface $v = \sqrt{4mr}$ as in Art. 82. Ex. 5.

Prop. CC.

If a chain whose length = 1 *be suspended at the top, it's lower end touching the earth, and then be let fall; to find the velocity of the chain.*

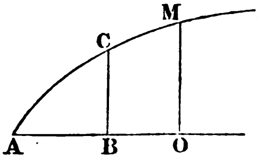
Every point of the chain being attracted to the earth by a greater force than the parts above, the lower parts will accelerate the parts above them; hence the chain will continue to be stretched, and the parts will continue to act on each other, just as if they were connected as in a rod. Hence, if $x = CD$ at any time, then $CO = \sqrt{rx}$, $\dot{C}O = \frac{1}{2} r^{\frac{1}{2}} x^{-\frac{1}{2}} \dot{x}$; also, $F = \dfrac{r}{x}$; hence, $v\dot{v} = -mr^{\frac{3}{2}} x^{-\frac{3}{2}} \dot{x}$, and $v^2 = 4mr^{\frac{3}{2}}\left(\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{c}}\right)$; hence, $v = \sqrt{4m} \times r^{\frac{3}{4}} \sqrt{\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{c}}}$.

If c be infinite, and the whole chain fall to the earth, then $x = r$, and $v = \sqrt{4mr}$. Hence it appears, that the greatest velocity which the chain can acquire is the same as that of a body falling from an infinite distance.

Prop. CCI.

Let ACZ *be a curve,* AB *the abscissa,* BC *the ordinate perpendicular to it; and suppose* ACZ *to be described by a body acted upon by a force perpendicular to* AB; *to find the law of force.*

Put $x = AB$, $y = BC$, $v =$ velocity in the direction CB, $F =$ force; then when the time is given, $F \propto \dot{v}$;



but $v \propto \dot{y}$, and $\dot{v} = \ddot{y}$; hence, $F \propto \ddot{y}$, $\dot{x}$ being given, since the force in the direction CB cannot change the velocity in the direction AB, or $\dot{x}$ is constant.

Ex. 1. Let ACZ be an ellipse, O the center, $OM = b$, $AO = a$, $\frac{b}{a} = c$; then $y^2 = c^2 \times (a^2 - x^2)$; hence, $y\dot{y} = -c^2 x\dot{x}$, and $y\ddot{y} + \dot{y}^2 = -c^2\dot{x}^2$, and $F \propto \ddot{y} \propto \frac{-c^2\dot{x}^2 - \dot{y}^2}{y} \propto \left(\text{as } \dot{y} = \frac{-c^2 x\dot{x}}{y}\right) \frac{-c^2\dot{x}^2 \times (y^2 + c^2x^2)}{y^3} = \frac{-c^2\dot{x}^2 \times c^2a^2}{y^3}$ which varies as $\frac{1}{y^3}$. The same is true for the hyperbola and parabola.

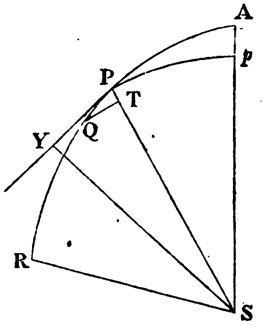
Ex. 2. For a parabola in general, $y^n = ax$, $ny^{n-1}\dot{y} = a\dot{x}$, and $\dot{x}$ being constant, $n.(n-1)\,y^{n-2}\dot{y}^2 + ny^{n-1}\ddot{y} = 0$; hence, $F \propto \ddot{y} \propto \frac{-n.(n-1)y^{n-2}\dot{y}^2}{ny^{n-1}} = \frac{-(n-1)\times\dot{y}^2}{ny} \propto \left(\text{as } \dot{y} = \frac{a\dot{x}}{ny^{n-1}}\right) \frac{-(n-1)a^2\dot{x}^2}{n^2y^{2n-1}} \propto \frac{1}{y^{2n-1}}$.

Prop. CCII.

If a body be compelled to move on a curve line APR *by a force tending to* S; *to find the velocity and time corresponding to any given space.*

Let the body begin to move from A, P any point, PY a tangent at P, SY a perpendicular to it; and take

PQ an indefinitely small part of the curve, and QT perpendicular to SP, and take $Sp=SP$. Put $SA=a$, $SP=Sp=x$, $AP=z$, $PQ=\dot{z}$, $PT=\dot{x}$, $v=$ velocity at P, $t=$time through AP, $F=$force at P in the direction PS.



Now (Art. 206.) the velocity beginning at A, the velocity in the curve at $P=$velocity in the straight line AS at p. Put $2m=32\frac{1}{6}$ feet representing the force of gravity at the earth's surface; then $v\dot{v}=-F\dot{x}$, the fluent of which gives v. Also $\dot{t}=\frac{\dot{z}}{v}$; but by sim. tri. $PY : x :: \dot{x} : \dot{z}=\frac{x\dot{x}}{PY}$; hence, $\dot{t}=\frac{x\dot{x}}{PY\times v}$, and the fluent gives the time.

Let $F\propto dx^n$; then (Pr. 41.) $v=\sqrt{\frac{4md}{n+1}}\times\sqrt{a^{n+1}-x^{n+1}}$; also $\dot{t}=\frac{1}{\sqrt{\frac{4md}{n+1}}}\times\frac{x\dot{x}}{PY\times\sqrt{a^{n+1}-x^{n+1}}}$, the fluent of which gives t.

Ex. Let APR be a parabola, S the focus, R the vertex; put $RS=b$, $n=-2$, then by conics $SY^2=bx$, and $PY=\sqrt{x^2+bx}$; hence, $\dot{t}=\frac{1}{\sqrt{4md}}\times$

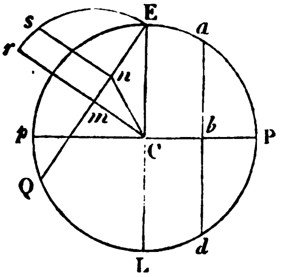
$$\frac{x\dot{x}}{\sqrt{x^2+bx}\times\sqrt{\frac{a-x}{ax}}}=\frac{a^{\frac{1}{2}}}{\sqrt{4md}}\times\frac{x\dot{x}}{\sqrt{ab+(a-b)\times x-x^2}}$$

$=$ (if $h=\frac{a^{\frac{1}{2}}}{\sqrt{4md}}$, $2c=a-b$) $\frac{hx\dot{x}}{\sqrt{ab+2cx-x^2}}$; put $x-c=y$, then (if $ab-c^2=e)^2$ $ab+2cx-x^2=y^2+e^2$, also $x\dot{x}=y\dot{y}+c\dot{y}$; hence, $\dot{t}=\frac{hy\dot{y}}{\sqrt{y^2+e^2}}+\frac{hc\dot{y}}{\sqrt{y^2+e^2}}$, and (Arts. 39, 45.) $t=h\sqrt{y^2+e^2}+hc\times$ h. l. $(y+\sqrt{y^2+e)^2}+C$; but when $t=0$, $x=a$, and $y=x-c$, hence, $t=h\sqrt{(x-c)^2+e^2}-h\times\sqrt{(a-c)^2+e^2}+hc\times$ h.l. $\frac{x-c+\sqrt{(x-c)^2+e^2}}{a-c+\sqrt{(a-c)^2+e^2}}$.

PROP. CCIII.

Let C *be the centre of the Earth,* EPLp *a great circle,* EL *a diameter,* EQ *any chord representing a perforation down which a body descends freely; to find the time of descent, and velocity.*

Draw Cm perpendicular to EQ, Er a quadrant of a circle about the centre m, sn perpendicular to EQ,



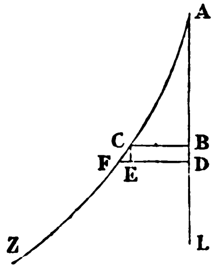
and PCp to ECL, and join Cn. Put $CE=r$ $Cm=e$, $mE=d$, $m=16\frac{1}{12}$ feet, and $2m$=gravity at E, $x=mn$,

v = velocity at n, t = time down En, $z = Er$; then $Cn = \sqrt{e^2 + x^2}$. Now $r : \sqrt{e^2 + x^2} :: 2m : \frac{2m}{r}\sqrt{e^2+x^2}$ the gravity at n in direction nC (Prop. 36. Cor. 2.); hence, $\sqrt{e^2 + x^2} : x :: \frac{2m}{r}\sqrt{e^2+x^2} : \frac{2mx}{r}$ the force in the direction nm; therefore $v\dot{v} = \frac{2m}{r} \times -x\dot{x}$, and the correct fluent gives $v = \sqrt{\frac{2m}{r}} \times \sqrt{d^2 - x^2}$, and when $x = 0$, $v = d\sqrt{\frac{2m}{r}}$ the velocity at m. If therefore a semicircle EmC be described on EC, the velocity down Em varies as Em. Also, $\dot{t} = \frac{-\dot{x}}{v} =$
$\frac{1}{\sqrt{\frac{2m}{r}}} \times \frac{-\dot{x}}{\sqrt{d^2 - x^2}} = \frac{1}{\sqrt{\frac{2m}{r}}} \times \frac{\dot{z}}{d}$, and $t = \frac{1}{\sqrt{\frac{2m}{r}}} \times \frac{\dot{z}}{d} =$ (when n comes to m) $\frac{1}{\sqrt{\frac{2m}{r}}} \times \frac{Es}{Em} = \frac{1}{\sqrt{\frac{2m}{r}}} \times \frac{EP}{r}$
$= \frac{EP}{\sqrt{2mr}}$ the time down which is therefore the same, in whatever direction EQ is drawn. Hence, the times of descent down all the chords EQ are equal. This also agrees with a constant force acting in a direction parallel to ECL.

PROP. CCIV.

If a body be projected at A *down a curve* AZ, *to find it's nature, so that the body may every where approach the horizon equally in equal times.*

Draw AL perpendicular to the horizon, and CB, FD, indefinitely near and parallel to it, also CE parallel to *BD*. Put $AB = x$, $BC = y$, $AC = z$, $CE = \dot{x}$, $CF = \dot{z}$, $a =$ space through which a body must fall to acquire the



velocity at *A*, $s =$ sine, $c =$ cosine of the angle *A*. Now the velocity at *C* is as $\sqrt{a+x}$ which is the velocity in the direction *CF*; hence, $\dot{z} : \dot{x} :: \sqrt{a+x} : \frac{\dot{x}\sqrt{a+x}}{\dot{z}}$ the velocity in the direction *CE*, a constant quantity; but at *A*, $\dot{z} : \dot{x} :: 1$ (rad.) $: c$, and $\frac{\dot{x}}{\dot{z}} = c$; also, $\sqrt{a+x}$ becomes $\sqrt{a}$; hence, $\frac{\dot{x}}{\dot{z}}\sqrt{a+x} = ca^{\frac{1}{2}}$, and $\dot{x}^2 \times (a+x) = c^2 a \dot{z}^2 = c^2 a \times (\dot{x}^2 + \dot{y}^2)$, and $\dot{x}\sqrt{s^2 a + x} = ca^{\frac{1}{2}}\dot{y}$, and the correct fluent is $\frac{2}{3} \times \overline{s^2 a + x}|^{\frac{3}{2}} - \frac{2}{3}s^3 a^{\frac{3}{2}} = ca^{\frac{1}{2}}y$ an equation to the semicubical parabola.

Prop. CCV.

Let $x^4 - ay^3 = my^2x$; *to find the area of the curve.*

Put $vy = x$, then $x^4 - av^3x^3 = mv^2x^3$, and $x = mv^2 + av^3$; hence, $y = mv^3 + av^4$; therefore $y\dot{x} = 2m^2v^4\dot{v} + 5mav^5\dot{v} + 3a^2v^6\dot{v}$, whose fluent is $\frac{2}{5}m^2v^5 + \frac{5}{6}mav^6 + \frac{3}{7}a^3v^7 = \frac{2}{5}m^2 \times \frac{y^5}{x^5} + \frac{5}{6}ma \times \frac{y^6}{x^6} + \frac{3}{7}a^2 \times \frac{y^7}{x^7}$. Now although when $x = 0$, $y = 0$, it does not necessarily follow that such a fluent wants no correction, because each term depends on the ultimate ratio of x to y, or on the angle which the curve first makes with the abscissa. In the present case we may consider, that as $x = mv^2 + av^3$, when $x = 0$, $v = 0$, m and a being positive numbers; and as $v = \frac{y}{x}$, $\frac{y}{x}$ ultimately becomes $= 0$, and the fluent wants no correction, which it would have done if v had been ultimately a finite quantity. This substitution may be successfully applied when one of the terms contains a power of x or y, and each of the other terms contains a power of x or y, such terms being one power lower in terms of x or y, or x and y together.

In EMERSON's Fluxions, Sect. II. Prop. 10. Ex. 23. the equation of the curve is $a^5 x^2 y^2 - x^9 = a^6 y^3$; and substituting $y = \frac{x^2}{z}$, the equation becomes $a^5 z - x^3 z^3 = a^6$; hence, getting x and y, he finds the fluent of $y\dot{x}$ to be $\frac{2a^5 y^3}{9x^6} - \frac{a^6 y^4}{4x^8}$, which he states to be the area. But this wants a correction, notwithstanding x and y vanish together; for from $a^5 z - x^3 z^3 = a^6$, when $x = 0$, $z = a$; hence, $\frac{x^2}{y} = z = a$ ultimately; therefore the above expression for the area becomes $-\frac{1}{36} a^2$; hence the correct fluent becomes $\frac{2a^5 y^3}{9x^6} - \frac{a^5 y^4}{4x^8} + \frac{1}{36} a^2$ the area required.

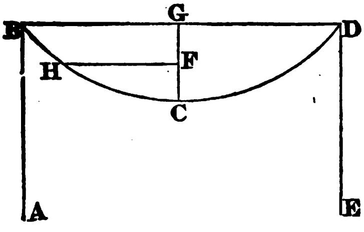
Let us take Ex. 24. where $x^3 + y^3 = axy$; and substituting $y = \frac{ax^2}{v^2}$ and finding x and y, he gets the fluent of $y\dot{x}$ to be $\frac{2ax^2}{3y} - \frac{x^4}{2y^2}$; but this also wants a correction; for $\frac{x^3}{y^3} = \frac{ax}{y^2} - 1$, and $\frac{x^6}{y^3} = \frac{ax^4}{y^2} - x^3$; now make $y=0$, then $x=0$, and $x^3=0$, and the other part of the last equation becomes $\frac{x^6}{y^3} = \frac{ax^4}{y^2}$, or $\frac{x^2}{y} = a$ ultimately; hence, the correct fluent becomes $\frac{2ax^2}{3y} - \frac{x^4}{2y^2} - \frac{1}{6}a^2 =$ the area.

The contents of solids generated by curves revolving about an axis may also be frequently found by a like substitution; and under similar forms of expression they will often want a correction.

Prop. CCVI.

If a uniform chain ABCDE *be suspended on two tacks* B, D, *in the same horizontal line, and* AB=DE; *to find when the chain will rest.*

Let $BG=GD$, and GC be perpendicular to BD, and HF to GC; put $CF=x$, $FH=y$, $CH=z$, $CG=d$, $BG=e$, $BC=s$; then (Prop. 131.) the weight of CH : tension at H :: $\dot{x}\left(\frac{z\dot{z}}{a+x}\right) : \dot{z}$ :: $z : a+x$; hence, the weight of CB : tension at B :: $s : a+d$; and if s, or BC, represent the weight of BC, $a+d$ represents the tension at B, or AB, the tension at B being represented by AB hanging freely down; hence, $2a+2d+2s=l$



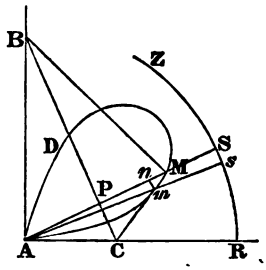
the whole length of the chain, and $2s = BCD$. Put c=number whose h. l. is 1; then (Prop. 131. Art. 109.) $c^{\frac{e}{a}} = \frac{s+\sqrt{s^2+a^2}}{a}$; and $s = \frac{a}{2}\left(c^{\frac{e}{a}} - c^{-\frac{e}{a}}\right) = BC$; also, $e = a \times \text{h. l. } \frac{a+d+\sqrt{d^2+2ad}}{a}$; and in like manner we find $a + d = a \times c^{\frac{e}{a}} - \sqrt{d^2+2ad} = a \times c^{\frac{e}{a}} - s = \frac{a}{2}\left(c^{\frac{e}{a}} + c^{-\frac{e}{a}}\right) = AB$.

Hence, $l = 2a \times c^{\frac{e}{a}}$; and to find the shortest chain possible that can be sustained, we must make the fluxion $=0$, or $\dot{a}c^{\frac{e}{a}} - \frac{ec^{\frac{e}{a}}\dot{a}}{a} = 0$; hence, $a = e$; put this for l, and $l = 2ac$, the shortest chain.

Prop. CCVII.

If the right-angled corner BAC *of a leaf be turned back into* BMC, *and moved into every position, the area* BAC (a) *remaining the same; to find the area of the curve* ADMA *described by the point* M.

Join AM, then APM is perpendicular to BC; draw Am indefinitely near to AM; with the centre A and



radius $AR=1$, describe the circular arc RZ, and produce AM, Am, to meet it in S, s. Put $AP = y$,

$RS = z$, $t =$ tan. of the angle RAS, then $\frac{1}{t}$ = tan. of BAP; and by sim. tri. $1 : t :: y : ty = PC$, $1 : \frac{1}{t} :: y : \frac{y}{t} = BP$; therefore $\left(ty + \frac{y}{t}\right) \times y = 2a$; hence $y^2 = \frac{2at}{1+t^2}$. Again, $1 : 2y :: \dot{z} : mn = 2y\dot{z}$; hence, the fluxion of the area $= 2y^2\dot{z} = \frac{4at\dot{z}}{1+t^2} = \left(\text{as } \dot{z} = \frac{\dot{t}}{1+t^2}\right) \frac{4at\dot{t}}{\overline{1+t^2}|^2}$, and the area $= \frac{-2a}{1+t^2} + C$; but when $t = 0$, the area $= 0$; hence, the correct fluent is $2a - \frac{2a}{1+t^2}$ = area $AMmA$; and when z=a quadrant, t is infinite, and the whole area $ADMA = 2a$.

PROP. CCVIII.

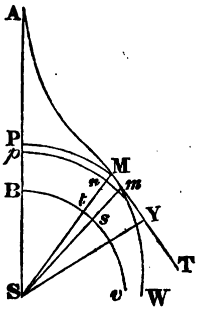
If a body fall towards a centre of force, z = *velocity due to the distance* x *from the centre,* d = *velocity at first,* v *and* a *the spaces through which a body must fall by gravity to acquire the velocities* z, d, *and* F = *force at the distance* x; *then* $v = a - \int F\dot{x}$.

For $v^2 = 4mz$, $d^2 = 4ma$, putting $m = 16\frac{1}{12}$ feet; and $v\dot{v} = -2mF\dot{x}$, and the fluent is $v^2 = 4m \times -\int F\dot{x}$; but at first, $v = a$; hence, $v^2 - a^2 = -4m\int F\dot{x}$, or $v^2 = a^2 - 4m\int F\dot{x}$, or $4mz = 4ma - 4m\int F\dot{x}$, or $z = a - \int F\dot{x}$.

If x = space fallen through, $z = a \pm 2mx$.

PROP. CCIX.

Let S *be the centre of force; to find the curve* AMW *in which if a body move, it shall approach the centre* S *equally in equal times.*



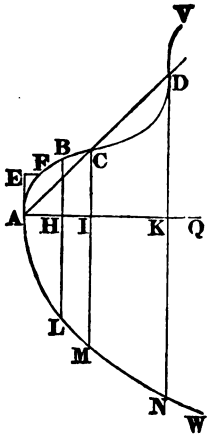
Let *SA* be a tangent to the curve at *A*, and *MY* at *M*, *SY* perpendicular to *MT*, *PM* a circular arc whose centre is *S*, *pm* another indefinitely near it. Let $p = SY$, the other letters as in the last Prop., then $z = a - \int F\dot{x}$. Now the velocity in *M* is such, that *Mm* is described in the same time with the velocity *z*, as *Pp* would be described with the velocity *d*; hence, $v : d$, or $\sqrt{z} : \sqrt{a} :: mM : Pp :: MS : MY$, therefore $a \times MS^2 = z \times MY^2 = a \times MY^2 - \int F\dot{x} \times MY^2$, and $a \times p^2 = (p^2 - x^2) \times \int F\dot{x}$, or $p^2 = \dfrac{x^2 \times -\int F\dot{x}}{a - \int F\dot{x}}$. With the centre *C* rad. $CB = 1$, describe the circle *Bv*, and let $s = \sin.\ ASM$, $u = Bt$. Now $Mm : mn :: x : p$, and $mn : ts\ (= \dot{u}) :: x : 1$; hence $\dot{u} = \dfrac{p \times Mm}{x^2}$; but $Mm = \dfrac{x\dot{x}}{My}$, = and $My = \dfrac{\sqrt{a} \times x}{\sqrt{z}}$; $\therefore Mm = \dfrac{x^2\sqrt{z}\dot{x}}{\sqrt{a}}$, and $\dfrac{Mm}{x^2} = \dfrac{\sqrt{z} \times \dot{x}}{\sqrt{a}}$; but $p = \dfrac{x\sqrt{-\int F\dot{x}}}{\sqrt{z}}$; hence $\dfrac{p \times Mm}{x^2} = \dot{u} = \dfrac{\dot{x}}{\sqrt{a} \times x} \times \sqrt{-\int F\dot{x}}$. If therefore *x* be given in terms of $\dot{u}$, and the fluent be found, we get *CM* in terms of the angle *ASM*.

If for F we put x, $-\int F\dot{x} = -\frac{1}{2}x^2 + C$; but $-\int F\dot{x} = z - a$, and at A, $z = 0$; that is, $-\int F\dot{x} = \frac{1}{2}\times(d^2 - x^2)$, and $\frac{1}{\sqrt{2}} \times \sqrt{d^2 - x^2} = \sqrt{-\int F\dot{x}}$; hence, $\dot{u} = \frac{1}{\sqrt{2a}} \times \frac{\dot{x}}{x}\sqrt{d^2 - x^2} = \frac{1}{\sqrt{2a}} \times \left(\frac{d^2\dot{x}}{x\sqrt{d^2 - x^2}} - \frac{x\dot{x}}{\sqrt{d^2 - x^2}}\right)$, whose fluent (Art. 39. 45.) is $u = \frac{1}{\sqrt{2a}} \times \sqrt{d^2 - x^2} + \frac{d}{2\sqrt{2a}} \times \text{h. l.}\ \frac{d - \sqrt{d^2 - x^2}}{d + \sqrt{d^2 - x^2}}$; hence, given CM, we know the angle ACM, and therefore the point M.

Prop. CCX.

Let AV *be a curve,* AH = x, HB = y, *take* AE (*parallel to* BH) = x, EF (*parallel to* AH) = z; *and let the ordinate* HL = z + y; *then if* $y = \overline{ax^m + bx^{m-1} + \&c.}|^{\frac{1}{p}}$, *to find the parts of the curve* ALW *which can be squared.*

Draw the straight line ACD making the angle DAQ of 45°, cutting the curve in C, D, &c. Now the



fluxion of $AEF = z\dot{x}$, and of $AHB = y\dot{x}$, and the sum of the two fluxions $= (y+z) \times \dot{x}$ the fluxion of AHL; therefore the area $AEFBH =$ area AHL; hence at C, the area $AMI =$ the square on AI, since $AI = CI$; same for AK, &c. &c. And at these points $z = x = y = \overline{ax^m + bx^{m-1} + \&c.}\Big|^{\frac{1}{p}}$, or $x^p - ax^m - bx^{m-1} - \&c. = 0$; let α, β, γ, &c. be the possible values of x in the last equation, then $\alpha = AI$, $\beta = AK$, &c. &c. and the area $AIM = \alpha^2$, $AKN = \beta^2$, &c. &c.

Prop. CCXI.

Given $y^n + ax^m y^r = bx^s$; *to express* x *and* y *in terms of a third variable quantity.*

Assume $y^n x^a = v^n$, then by substitution $v^n + \frac{av^r}{x^{\frac{ar-mn-an}{n}}} = bx^{s+a}$; make $\frac{ar-mn-an}{n} = 0$, or $a = \frac{mn}{r-n}$; then $v^n + av^r = bx^{s+a}$, and $x = \overline{\frac{v^n+av^r}{b}}\Big|^{\frac{1}{s+a}}$; also, $y^n = \frac{v^n}{x^a} = \frac{v^n}{\overline{\frac{v^n+av^r}{b}}\Big|^{\frac{a}{s+a}}}$.

Hence, we get $y\dot{x}$ in terms of v and it's fluxions, and therefore when we can find the fluent, we can find the area of the curve. In like manner, when we can find the fluent of $y^2\dot{x}$, we can find the solid generated by the revolution of the curve about it's axis; and so on, for other quantities.

Prop. CCXII.

Given the fluents of $y\dot{x}$, $yx\dot{x}$, $yx^2\dot{x}$ ..... $yx^{n-1}\dot{x}$; *and let* $y\dot{x} = \dot{A}$, $A\dot{x} = \dot{B}$, ... $Q\dot{x} = \dot{R}$, $R\dot{x} = \dot{S}$, $Sx = \dot{T}$, *of which the number is* n; *to find the fluent of* $\dot{T}$.

As $S\dot{x} = \dot{T}$, let $Sx - a = T$, then $S\dot{x} + x\dot{S} - \dot{a} = \dot{T}$, and hence, $\dot{\alpha} = x\dot{S} = Rx\dot{x}$; let $\frac{Rx^2}{2} - \beta = \alpha$, then $Rx\dot{x} + \frac{x^2}{2}\dot{R} - \dot{\beta} = \dot{\alpha}$, therefore $\dot{\beta} = \frac{x^2}{2}\dot{R}$; $= Q\frac{x^2}{2}\dot{x}$; let $Q\frac{x^3}{2.3} - \gamma = \beta$, then $Q\frac{x^2\dot{x}}{2} + \frac{x^3}{2.3}\dot{Q} - \dot{\gamma} = -\dot{\beta}$, and $\dot{\gamma} = \frac{x^3}{2.3}\dot{Q}$; proceed thus, and we get $T = Sx - R\frac{x^2}{2} + Q\frac{x^3}{2.3} -$ &c. $\pm \frac{1}{1.2\ldots(n-1)} \times Ax^{n-1} - \int yx^{n-1}\dot{x}$.

Prop. CCXIII.

If a body fall from rest in a right line towards the centre of force; to find in what cases the whole time of descent can be exhibited in finite terms.

If $A = \frac{1}{\sqrt{\frac{4md}{n+1}}}$, then (Art. 83.) $\dot{t} = -A \times \frac{\dot{x}}{\sqrt{a^{n+1} - x^{n+1}}}$; let the force vary as $x^{\frac{1-m}{m}}$; then for n substitute $\frac{1-m}{m}$, and $\dot{t} = -A \times \frac{\dot{x}}{\sqrt{a^{\frac{1}{m}} - x^{\frac{1}{m}}}}$. Put $d = a^{\frac{1}{m}}$, $z = x^{\frac{1}{m}}$; then $\dot{t} = -A \times \frac{mz^{m-1}\dot{z}}{\sqrt{d-z}}$. Let $\sqrt{d-z} = y$, then $z = d - y^2$, and $mz^{m-1}\dot{z} = -2m \times \overline{d-2y^2}|^{m-1}y\dot{y}$; hence, $\dot{t} = 2A[d^{m-1}\dot{y} - (m-1).d^{m-2}y^2\dot{y} + (m-1) \times \frac{m-2}{2}.d^{m-3}y^4\dot{y} -$ &c.$]$, and $t = 2A\,(dy^{m-1} - \frac{m-1}{1.3}d^{m-2}y^3 + \frac{(m-1).(m-2)}{1.3.5}d^{m-3}y^5 -$ &c.$)$ Now when $x=0$, $z=0$, and $y=d^{\frac{1}{2}}=a^{\frac{1}{2m}}$; hence, $t=2Aa^{\frac{2m-1}{2m}}\cdot\left(1-\frac{m-1}{1.3}+\frac{(m-1).(m-2)}{1.3.5}-\&c.\right)$ the whole time.

If $m=1$, $t=2Aa^{\frac{1}{2}}$

$m=2$, $t=2Aa^{\frac{3}{4}}\times\frac{2}{3}$,

$m=3$, $t=2Aa^{\frac{5}{6}}\times\frac{2.4}{3.5}$

in general, $t=2Aa^{\frac{2m-1}{2m}}\times\frac{2.4.6\ldots.2m-2}{3.5.7\ldots.2m-1}$.

In like manner, there are other quantities which, substituted for n, will give the time in finite terms.

Prop. CCXIV.

Given the time of descent in a right line towards the centre of force; to find the force F.

If a = whole distance, x any variable distance; then $v\dot{v}=-F\dot{x}$, and $v=\sqrt{2\int-F\dot{x}}$; also, $\dot{t}=\frac{-\dot{x}}{\sqrt{2\int-F\dot{x}}}$; hence, $\sqrt{2\int-F\dot{x}}=\frac{\dot{x}^2}{\dot{t}^2}$; let $\dot{x}$ be constant, then $-2F\dot{x}=\frac{\dot{x}^2\ddot{t}}{\dot{t}^3}$, and $F\propto\frac{\dot{x}\ddot{t}}{\dot{t}^3}$.

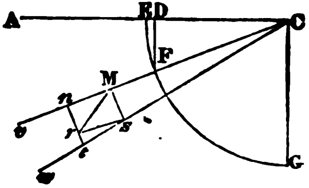
Let $t\propto\sqrt{a-x}$, then $t^2\propto a-x$, $2t\dot{t}\propto-\dot{x}$, and $2\dot{t}^2+2t\ddot{t}=0$; hence, $F\propto 1$, a constant quantity.

Let $t\propto\overline{a-x}|^{n}$, then $F\propto\frac{1}{\overline{a-x}|^{2n-1}}$.

Prop. CCXV.

Let a ring at C *be put on an horizontal rod* AC, *moveable about* C *uniformly in a vertical plane; to find the place* M *of the ring at any time.*

With the centre C and radius $CE=1$, describe the



quadrantal arc EG; draw FD perpendicular to CA, and Cw indefinitely near Cv; Ms, nt perpendicular to Cv and sr to Ms. Put $CE=d$, $CM=x$, $EF=z$, $DF=y$, $DC=\pi$, $m=16\frac{1}{12}$ feet, $v=$ velocity of the ring at M in the direction of the rod, $t=$ corresponding time, $b=$ arc of EG described in $1''$; then $1 : x :: b : bx=$ the velocity of the ring at M perpendicular to the rod; hence, (Art. 202.) $b^2x=$ the centrifugal force at M; also, $2my=$ the force of gravity down the rod; hence, $b^2x+2my=$ the whole force urging the ring down the rod. Again, $b : \dot{z} :: 1'' : \dot{t}=\frac{\dot{z}}{b}$; also, $\dot{t}=\frac{\dot{x}}{v}$; hence, $v=\frac{b\dot{x}}{\dot{z}}$, and $\dot{v}=\frac{b\ddot{x}}{\dot{z}}$, $\dot{z}$ being constant. Now $(b^2x+2my)\times\frac{\dot{z}}{b}=\dot{v}=\frac{b\ddot{x}}{\dot{z}}$, and if $n=\frac{2m}{b^2}$, $\frac{\ddot{x}}{\dot{z}^2}-x-ny=0$. Assume $x=pz^2+qz^3+rz^4+sz^5+uz^6+wz^7+$ &c. Then $\frac{\ddot{x}}{\dot{z}^2}=1.2p+2.3qz+3.4rz^2+4.5sz^3+5.6uz^4+6.7wz^5+$ &c.; also (Prop. 128.) $y=z-\frac{z^3}{2.3}+\frac{z^5}{2.3.4.5}-$ &c. hence, by substitution,

$$\left.\begin{array}{l} 1.2p+2.3qz+3.4rz^2+4.5sz^3+5.6uz^4+6.7wz^5+\&c. \\ \qquad\qquad -pz^2-\quad qz^3-\quad rz^4-\quad sz^5-\&c. \\ \quad -nz \qquad +\frac{n}{2.3}z^3 \quad -\frac{n}{2.3.4.5}z^5+\&c. \end{array}\right\}=0,$$

therefore (Art. 110.) $p=0$, $2.3q-n=0$, $3.4r-p=0$, $4.5s-q+\frac{n}{2.3}=0$, $5.6u-r=0$, $6.7w-s-\frac{n}{2.3.4.5}=0$, &c. hence, $q=\frac{n}{2.3}$, $r=0$, $s=0$, $u=0$, $w=\frac{n}{2.3.4.5.6.7}$, &c. and $x=n\left(\frac{z^3}{2.3}+\frac{z^7}{2.3.4.5.6.7}+\&c.\right)$; but (Prop. 50. Cor.) $\frac{a^z-a^{-z}}{2}=z+\frac{z^3}{2.3}+\frac{z^5}{2.3.4.5}+\frac{z^7}{2.3.4.1.6.7}+$ &c. hence, $x=\frac{m}{b^2}\cdot\left(\frac{a^z-a^{-z}-zy}{2}\right)$. Corresponding therefore to z, or to $\frac{z}{b}$, we know x.

As $v=\frac{b\dot{x}}{\dot{z}}$, we have $v=\frac{m}{b}\left(\frac{a^z+a^{-z}-2\pi}{2}\right)$.

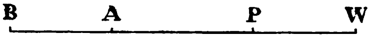
If we neglect the centrifugal force, the equation becomes $\frac{\ddot{x}}{\dot{z}^2}=ny$, or $\frac{\ddot{x}}{\dot{z}}=ny\dot{z}=-n\dot{\pi}$, whose fluent is $\frac{\dot{x}}{\dot{z}}=-n\pi$, and $\dot{x}=-n\pi\dot{z}=-n\dot{y}$, and the correct fluent is $x=n\times(z-y)=\frac{2m}{b^2}\times(z-y)$. Also, $v=\frac{b\dot{x}}{\dot{z}}=\frac{2m}{b}\left(\frac{\dot{z}}{\dot{z}}-\frac{\dot{y}}{\dot{z}}\right)=\frac{2m}{b}(1-\pi)=\frac{2m}{b}\times DE$. The same conclusion may be obtained thus: $\dot{t}=\frac{\dot{z}}{b}$; but $\dot{z}=\frac{\dot{y}}{\sqrt{1-y^2}}$; and the accelerative force $=2my$; hence, $\dot{v}=2my\times\frac{\dot{z}}{b}=\frac{2m}{b}\times\frac{y\dot{y}}{\sqrt{1-y^2}}$, and $v=\frac{2m}{b}\times(-\sqrt{1-y^2}+C)=\frac{2m}{b}\times(1-\pi)$.

If Ms represent bx, and Mn represent v; then by these two velocities continued uniform, the ring at M would be carried to r, and thus the perpendicular velocity would be *retarded* by rt, or by $b\dot{x}$; but as the perpendicular velocity of M is represented by Ms, or bx, it's fluxion $b\dot{x}$ expresses how much that velocity is accelerated; hence, $2b\dot{x}$, or $2v\dot{z}$, express the fluxion of the circulatory velocity arising from the excess of gravity above the pressure P on the rod. Now $2m\pi$ expresses the force of gravity; hence, $2m\pi - P =$ the whole force perpendicular to the rod; therefore $2v\dot{z} = (2m\pi - P) \times \frac{\dot{z}}{b}$, and $P = 2m\pi - 2vb \doteq m\{4\pi - (a^z + a^{-z})\}$. When $P = 0$, $4\pi - (a^z + a^{-z}) = 0$, and $\pi = \frac{a^z + a^{-z}}{4}$ 0,082377 answering to 47°. 12′; at which point, if the ring were not attached to the rod, it would quit it.

Prop. CCXVI.

Let BAPW *be an indefinite straight line*, BA = a, BP = x, *and a body* (A) *set out from* A, *and the velocity at* P *be as* x^n; *to find the time of describing* AP.

Let $t =$ time, $v =$ velocity at A; then $a^n : x^n :: v :$



$\frac{v}{a^n} \times x^n$ the velocity at P; hence, $\dot{t} = \frac{\dot{x}}{v} = \frac{a^n}{v} \times x^{-n}\dot{x}$, and corrected, $t = \frac{a^n}{v} \times \frac{x^{1-n} - a^{1-n}}{1-n}$.

If n be less than 1, when x is infinite, t is infinite.

If $n = 1$, $t = \frac{a}{v} \times$ h. l. $\frac{x}{a}$; and when x is infinite, t is infinite.

If n be greater than 1, $t = \frac{a^n}{v \times (n-1)} \left(\frac{1}{a^{n-1}} - \frac{1}{x^{n-1}}\right)$; and when x is infinite, $t = \frac{a}{v \times (n-1)}$ is finite. In all such cases therefore, *an infinite space would be described in a finite time.* If $a = 10$ feet, $v = 10$ feet, $n = 2$, then $t = 1''$ in which time the space AW is infinite.

PROP. CCXVII.

Let the axis of an heavy circle passing through it's centre be perpendicular to it's plane, about which let it revolve standing on an horizontal plane; to find it's centrifugal force in terms of it's weight.

Let w = weight of the circle, r = it's radius, v = velocity of it's circumference, x = any part of a radius from the centre, $2m = 32\frac{1}{6}$ foot, $p = 6{,}283$, &c. a = velocity in falling through $\frac{1}{2}r$ by gravity; then (Prop. 149.) this velocity is such as would give a centrifugal force equal to it's gravity, or weight of a particle d; hence, $a^2 : v^2 :: d : d\frac{v^2}{a^2}$ the centrifugal force with the velocity v; and $r : x :: d\frac{v^2}{a^2} : \frac{dv^2}{ra^2} \times x$ the centrifugal force at the distance x from the centre; or instead of d, putting $px\dot{x}$ for the fluxion of the circle at the distance x, we have $\frac{pv^2}{ra^2} \times x^2\dot{x}$ the fluxion of the centrifugal force, whose fluent (when $x = r$) is $\frac{pv^2r^2}{3a^2}$ the centrifugal force of the whole circle. But the fluxion of the circle (or of it's weight) being represented by $px\dot{x}$, w will be represented by $\frac{pr^2}{2}$; also, $a^2 = 2mr$; hence, the centrifugal force will be represented by $\frac{wv^2}{3mr} = \frac{4}{193} \times \frac{wv^2}{r}$.

Ex. Let $w = 4$ oz. $v = 12$ feet, $r = \frac{1}{12}$ foot; then the centrifugal force $= 143\frac{1}{4}$ oz. $= 11\frac{15}{16}$ of a lb., the whole force exercised on the axis.

Hence the reason why a whirligig gives so great a resistance to the touch, when the finger is applied to the axis whilst it is revolving. And this appears to be the reason why, when you want to catch on a point, a ball drawn up by a string, you first twist the string to give it a rotatory motion, because it's axis is then less likely to get out of it's perpendicular position from the action of the string.

Prop. CCXVIII.

If a spring be fixed at it's upper end in a perpendicular position, and be drawn through a small space (a) *out of that position by a weight* (Q) *fixed by a string to it's lower end acting horizontally against it; then upon the removal of* Q, *let the lower end impel a body* (P) *along an horizontal plane; to find the velocity of* P, *and time of describing any space.*

Let $x =$ any space described from the quiescent state of P, $v =$ velocity, $t =$ time, $m = 16\frac{1}{12}$ feet. By the property of the spring $a : x :: Q : \frac{Q}{a} \times x$ the moving force; now this force lying in the spring, it has to move P and the spring also. Let the weight of the spring $= w$, and nw be a weight placed at the lower end such, that it's inertia may be equivalent to that of the spring, where n must be determined by experiment.

Then the accelerative force of $Q = \dfrac{\frac{Q}{a} \times x}{P + nw} = \dfrac{Qx}{Pa + naw}$

hence, $v\dot{v} = \dfrac{2mQ}{Pa + naw} \times -x\dot{x}$, whose correct fluent gives $v = \sqrt{\frac{2mQ}{Pa + naw}} \times \sqrt{a^2 - x^2}$. Let $z =$ a quadrant of a circle whose radius is a, cosine x, $p =$ 1,57079. Now $\dot{t} = \frac{-\dot{x}}{v} = \sqrt{\frac{Pa+naw}{2mQ}} \times \frac{-\dot{x}}{\sqrt{a^2-x^2}} = \sqrt{\frac{Pa + naw}{2mQ}} \times \frac{\dot{z}}{a}$, and $t = \sqrt{\frac{Pa + naw}{2mQ}} \times \frac{z}{a} =$ (when $x = 0$, $z = pa$) $\sqrt{\frac{Pa + naw}{2mQ}} \times p$. Since Q varies as a, all the times of describing the whole spaces are the same, whatever a may be.

When $x = 0$, $v = \sqrt{\frac{2mQa}{p + nw}}$ the greatest velocity; if therefore we determine v by experiment, we get n.

Prop. CCXIX.

Let a fine thread be closely wound round a cone from the vertex, and turned about it's axis lying horizontally, by a weight P *hanging from the end of the thread next the base; to find the velocity and time of descent.*

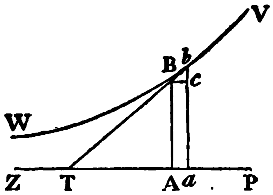
The thread is supposed so fine, that each round may be considered as perpendicular to the axis. Let $r =$ the distance of the center of gyration from the axis, $R =$ radius of the base, then $r = R\sqrt{\frac{3}{10}}$; $W =$ weight of the cone, $a =$ radius from which P at first hangs, $x =$ any other radius, $v =$ corresponding velocity, $n =$ number of rounds corresponding to a diminution d of the radius, $p = 6{,}283$ &c. Then $d : a - x :: n : \frac{n}{d} \times (a - x)$ the number of rounds between the radii a, x; and as we may consider $p \times \frac{a+x}{2}$ = the mean of these rounds, the whole quantity of thread run off, or the descent of P, is $\frac{pn}{2d} \times (a^2 - x^2)$; also (Art. 60.) $W \times \frac{r^2}{x^2}$ = quantity of matter which equally dispersed round the circumference of which x is the radius, is equivalent to the inertia of the cone; hence, the accelerative force $= \frac{P}{P + W\frac{r^2}{x^2}}$; also, the fluxion of the space described by $P = \frac{pn}{d} \times - x\dot{x}$; hence, $v\dot{v} = \frac{2mpnP}{d} \times \frac{-x\dot{x}}{P + W\frac{r^2}{x^2}} = \left(\text{putting } g = \frac{2mpn}{d},\ \pi = \frac{W}{P}\right) g \times \frac{-x^3\dot{x}}{x^2 + \pi r^2} = -gx\dot{x} + g\pi r^2 \times \frac{x\dot{x}}{x^2 + \pi r^2}$, whose correct fluent gives $v = \sqrt{g \times (a^2 - x^2) + g\pi r^2 \times \text{h. l. } \frac{x^2 + \pi r^2}{a^2 + \pi r^2}}$. The time cannot be found in general.

If we consider the cone as without weight, $v = \sqrt{g \times (a^2 - x^2)}$. This velocity therefore follows the same law as in Ex. 2. Art. 83, where a is the whole space to be described, and x any variable space. But by that Art. if $x = 0$, the times of descent are all equal, whatever be the value of a; hence, whatever be here the value of a, the time of unrolling the whole will be the same, provided g be the same, or $n : d$ in a given ratio; and that time $= \frac{1}{\sqrt{g}} \times \frac{1}{4}p = \sqrt{\frac{pd}{32mn}}$.

PROP. CCXX.

Let a body B *be drawn at* T *by a string* TB, T *moving in a straight line* PZ; *to find the curve* VBW *described by* B.

Put the abscissa $PA = x$, ordinate $AB = y$, $b = TB$,



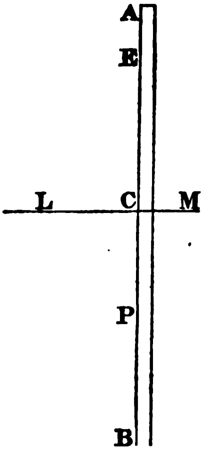
then TB must be always a tangent at B; and by sim. tri. $-\dot{y}\ (bc) : \dot{x}\ (Bc) :: y : \sqrt{b^2 - y^2}$, and $\dot{x} = \frac{-\dot{y}}{y} \times \sqrt{b^2 - y^2} = \frac{-b^2\dot{y} + y^2\dot{y}}{y\sqrt{b^2 - y^2}} = \frac{-b^2 y^{-1}\dot{y}}{\sqrt{b^2 - y^2}} + \frac{y\dot{y}}{\sqrt{b^2 - y^2}}$, whose fluent (Prop. 18. Ex. 8. and Prop. 16.) is $x = b \times \text{h. l.}\ \frac{b + \sqrt{b^2 - y^2}}{by} - \sqrt{b^2 - y^2} + C$; but supposing the abscissa to begin at the point where the ordinate AB is a tangent to the curve, when $x = 0$, $y = b$; and the correct fluent is $x = b \times \text{h. l.}\ \frac{b + \sqrt{b^2 - y^2}}{y} - \sqrt{b^2 - y^2}$. This is called the curve of *Traction*. T may be drawn along a curve line, but then the solution becomes more complex.

PROP. CCXXI.

Let LCM *be the surface of a fluid,* AB *a tube filled with the fluid and closed at* A *by the finger; to find the point* P *to which the fluid will descend upon removing the finger.*

Let $AB = a$, $CB = 1$, $AE = x$, $v =$ velocity at E,

$m = 16\frac{1}{12}$ feet, and let $2ma$ = gravity or weight of



the fluid in AB, $2m \times 1 = 2m$ = the gravity in CB, and $2m \times CE$ = gravity in $CE = 2m \times (a-1-x)$, and this is the moving force; therefore the accelerative force $= \frac{2m \times (a-1-x)}{a-x}$; hence, $v\dot{v} = \frac{2m \times (a-1-x)}{a-x} \times \dot{x} =$

$2m\left(\dot{x} - \frac{\dot{x}}{a-x}\right)$ and $v^2 = 4m\,[x + \text{h. l. } (a-x) + C]$;

but when $v = 0$, $x = 0$, and the correct fluent is $v^2 = 4m\,[x + \text{h. l. } a + \text{h. l. } (a-x)]$; and the lowest point P is found by making $v = 0$, or $x = \text{h. l. } a - \text{h. l. } (a-x)$, and x may be readily found by trial, by a table of hyp. logs.

PROP. CCXXII.

Let $\dot{y} + Py\dot{x} = Q\dot{x}$; *to separate the variable quantities,* P *and* Q *being functions of* x; *and thence to find the fluent when it can be found.*

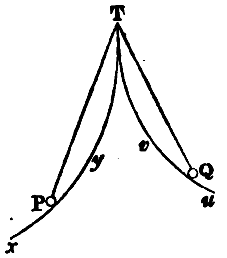
Let $y = vu$, then $\dot{y} = v\dot{u} + u\dot{v}$, and $v\dot{u} + u\dot{v} = Q\dot{x} - Pvu\dot{x}$; assume $\dot{v} + Pv\dot{x} = 0$, then $\frac{\dot{v}}{v} = -P\dot{x}$, and log. $v = -\int P\dot{x}$, and (Art. 111, Cor.) $v = a^{-\int P\dot{x}}$; but from the assumption, $v\dot{u} = Q\dot{x}$, $u = \int \frac{Q\dot{x}}{v} = \int a^{\int P\dot{x}} Q\dot{x} = \frac{y}{v}$; hence, $y = a^{-\int P\dot{x}} \int a^{\int P\dot{x}} Q\dot{x}$, and the quantities x and y are separated; and according to the functions of P and Q in terms of x, the fluent may or may not be found. For this we are indebted to *James Bernoulli.*

For more on the subject of Fluxional Equations, see *Waring's Meditationes Analyticæ.*

Prop. CCXXIII.

Let two bodies P, Q, *connected by a string going over a pulley at* T, *and lying upon two curves* xy, vw, *and* P *to descend; to find the velocities of* P *and* Q, *after* P *has descended perpendicularly through any given space**.

Let x = perpendicular descent of P, y = that of the ascent of Q, and at any time let P acquire the real velocity v, and Q that of w; then (Prop. 146.) the



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* We have here given the proposition, as being of a fluxional nature, and sometimes also in it's application.

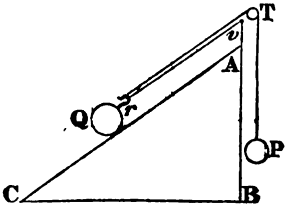
moving force of $P = \frac{Pv\dot{v}}{2m\dot{x}}$ in the direction of gravity; but when unconnected with the machine, P is the moving force; hence, by it's connection with Q, in moving force P has lost $P - \frac{Pv\dot{v}}{2m\dot{x}}$ in the direction of gravity. For the same reason, $Q + \frac{Qw\dot{w}}{2m\dot{y}}$ ($\dot{y}$ having a contrary sign to that of $\dot{x}$) is the motive force lost by Q; and (Mech. Art. 149.) $\dot{x} : \dot{y} :: Q + \frac{Qw\dot{w}}{2m\dot{x}} : \frac{Q\dot{y}}{\dot{x}} + \frac{Qw\dot{w}}{2m\dot{x}}$ an equivalent motive force at P; it must therefore be this force which destroys the motive force $P - \frac{Pv\dot{v}}{2m\dot{x}}$ lost by P; hence $P - \frac{Pv\dot{v}}{2m\dot{x}} = \frac{Q\dot{y}}{\dot{x}} + \frac{Qw\dot{w}}{2m\dot{x}}$, and $2m \times (P\dot{x} - Q\dot{y}) = Pv\dot{v} + Qw\dot{w}$, whose fluents are $4m \times (Px - Qy) = Pv^2 + Qw^2$. Now if $\dot{s}$ = the real space described by P, and $\dot{S}$ that described by Q in the time $\dot{t}$, then $\dot{t} = \frac{\dot{s}}{v} = \frac{\dot{S}}{w}$, and $w = \frac{v\dot{s}}{\dot{S}}$; hence, $4m \times (Px - Qy) = Pv^2 + Q \times \frac{v^2\dot{s}^2}{\dot{S}^2}$, and $v = \sqrt{\frac{4m \times (Px - Qy)}{P + Q \times \frac{\dot{s}^2}{\dot{S}^2}}}$. The velocity being determined in terms of the space, the time is found from $\dot{t} = \frac{\dot{x}}{v}$.

If c = cos. TPy, C = cos. TQv, then $\frac{\dot{s}}{\dot{S}} = \frac{C}{c}$, and $v = \sqrt{\frac{4m \times (Px - Qy)}{P + Q \times \frac{C^2}{c^2}}}$.

This proposition is to be applied in all cases where the velocities of the two bodies are not equal.

Ex. 1. *Let* AC (=a) *be an inclined plane,* Q *a weight connected to* P *by a string moving over a pulley at* T, *passing under a hook at* r *fixed to* Q *and returns and is fixed at* v, *the string acting parallel to* AC, *and* P *hanging freely down ; to find the velocity of* P *descending down the space* x.

Here x lengthens as the two parts vr, rT, shorten, therefore v being the velocity of P, $w = \frac{1}{2}v$; now when



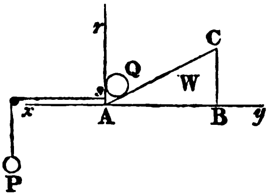
Q has ascended from C to A, or perpendicularly through BA $(=b)$, P has descended perpendicularly through $2a$; therefore $2a : b :: x : \frac{bx}{2a}$ the perpendicular ascent y of Q corresponding to the perpendicular descent x of P; hence, $v = 2\sqrt{mx} \times \sqrt{\frac{P - Q \times \frac{b}{2a}}{P + C^2 \times Q}}$. Also, the accelerative force $= \frac{v^2}{4mx} = \frac{P - Q \times \frac{b}{2a}}{P + C^2 \times Q}$; hence the corresponding time $= \sqrt{\frac{x}{mF}} = \sqrt{\frac{x}{m}} \times \sqrt{\frac{P + C^2 \times Q}{P - Q \times \frac{b}{2a}}}$.

If AC be horizontal, a is infinite, and $T = \sqrt{\frac{x}{m}} \times \sqrt{\frac{4P + Q}{4P}}$.

If Q hang perpendicularly down, $C=1$, and $v = \sqrt{\frac{4m \times (Px - Qy)}{P + Q \times \frac{1}{c^2}}}$.

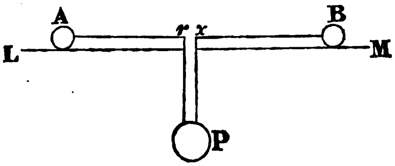
Ex. 2. *Let* P *hanging freely down, draw a wedge* ABC *along an horizontal plane, having on it a weight* Q *supported by a perpendicular plane* sr, *so that* Q *may ascend perpendicularly; to find the velocity of* P.

Let $AB : BC :: 1 : r$, then if P descend through x, Q ascends through rx; and if $v =$ velocity of P, $rv =$ velocity of Q; hence, $4m \times (Px - Qrx) = (P + W) \times v^2 + Q \times r^2 v^2$ (for P and W move with the same velocity); hence, $v = \sqrt{\frac{4m \times (Px - Qrx)}{P + W + Qr^2}}$. Hence we get the time as before.



Ex. 3. *Let* P *draw* A, B, *along the horizontal plane* LM, *hanging freely down, and connected by a string running through* P; *to find the velocity of* P.

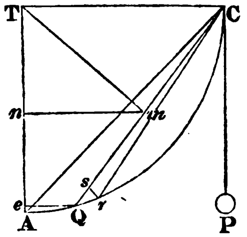
Let $x =$ space descended through by P, v it's velocity; then P acting on each body, the velocities or spaces passed over by each body will be inversely as their quantities of matter, and as the perpendicular parts of



the string have each lengthened by x, the two horizontal parts have together shortened by $2x$; but the vel. A : vel. B :: B : A :: $\frac{2Bx}{A+B} : \frac{2Ax}{A+B}$; A has therefore shortened by $\frac{2Bx}{A+B}$, B by $\frac{2Ax}{A+B}$; hence, the velocities of P, A, B, are v, $\frac{2Bv}{A+B}$, $\frac{2Av}{A+B}$; therefore $4m \times Px = Pv^2 + \frac{4B^2Av^2}{(A+B)^2} + \frac{4A^2Bv^2}{(A+B)^2}$; hence, $v =$
$$\frac{2\sqrt{mPx} \times (A+B)}{\sqrt{P \times (A+B^2) + 4B^2A + 4A^2B}}.$$
The time is found as before.

Ex. 4. *Let* C *be the center of the quadrantal arc* TA, CT *horizontal,* P *hanging freely down and connected with* Q *by a string passing over a pulley at* C, *drawing* Q *up the curve from* A; *to find* P's *velocity.*

Join CQ, bisect it at m, draw Cr indefinitely near CQ, and draw Qe, mn perpendicular to AT,



and rs to CQ. Put $r = CA$, $y = Ae$; then $eQ = \sqrt{2ry - y^2}$, and mn $(= \frac{1}{2}TC + \frac{1}{2}eQ) = \frac{1}{2}r + \frac{1}{2}\sqrt{2ry - y^2}$, and $Cn = \frac{1}{2}r - \frac{1}{2}y$; hence, $Cm = \sqrt{\frac{1}{2}r^2 + \frac{1}{2}r\sqrt{2ry - y^2}}$; and by sim. triangles $\dot{s}$ (Qs) :

$\dot{S}\,(Qr) :: \sqrt{\tfrac{1}{2}r^2 + \tfrac{1}{2}r\sqrt{2ry - y^2}} : r$; and $\frac{\dot{s}}{\dot{S}} = \frac{\sqrt{\frac{1}{2}r^2 + \frac{1}{2}r\sqrt{2ry - y^2}}}{r}$ hence; $v =$

$$\sqrt{\frac{4m \times (Px - Qy)}{P + Q \times \frac{\frac{1}{2}r^2 + \frac{1}{2}r\sqrt{2ry - y^2}}{r^3}}}.$$

And when Q comes to C, if $l =$ the quadrantal chord CA, then $x = l$, $y = r$, and $v = \sqrt{\frac{4m \times \overline{Pl - Qr}}{P + Q}}$ the velocity of P at that point. And this is the same velocity as that which would be acquired by P drawing Q up the inclined plane AC.

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Corrections of typographical ERRATA *in Professor* VINCE'S FLUXIONS.

PAGE 27. l. 1. *for* $+ 6x^4\dot{x}$ *read* $- 6x^4\dot{x}$. 34. l. 15. *f.* $2\sqrt{2a}$ *r.* $2a\sqrt{2}$. 38. l. 2. *f.* AE *r.* DE. 40. l. 21. *f.* $\sqrt{m-x^3}$ *r.* $\sqrt[3]{m-x^3}$. 48. l. 4. *f.* 130 *r.* 131. 58. l. 1. *f.* BL *r.* PL; l. 2, 7 *f.* PT *r.* PL. 61. l. 4. *f.* x *r.* $\dot{x}$. 86. l. 23 *f.* 130 *r.* 131. 88. l. 18. *f.* $x^3\dot{x}$ *r.* $x^2\dot{x}$. 96. l. 2. *f.* m *r.* w; l. 5, 8. *f.* cx *r.* pcx. 98. l. 3. *f.* 130 *r.* 131; l. 12. *f.* ay^2x *r.* py^2x. 106. l. 20. *f.* d^2 in den. *r.* a^2. 113. l. 5. *f.* 130 *r.* 131; l. 20. *f.* D *r.* Q; in fig. *f.* v *r.* q. 119. l. 16. *f.* bisects *r.* is bisected by. 121. l. 6. *f.* 59 *r.* 60. 129. l. 21. *f.* EGF *r.* HGF. 131. l. 18. *f.* $3.3(n+1)$ *r.* $3(3n+1)$. 136. l. 1. *f.* $\frac{2}{3}mn(b-a)$ *r.* $\frac{2}{3}mn^3(b-a)$. 142. l. 12. *f.* Pr *r.* PR. 144. l. 2. *f.* $PK^2 - \overline{c+x}|^2$ *r.* $PK^2 = \overline{c+x}|^2$. 146. l. 11. *f.* A *r.* a. 155. l. 18. *f.* $t^{2m} + m^2y^{2m}$ *r.* $(t^{2m} + m^2y^{2m})^2$. 168. l. 6. *f.* ev^2 *r.* ev^c. 172. l. 19. *f.* $3F\dot{z}$ *r.* $2F\dot{z}$. 173. l. 5. *f.* 54 *r.* 98. 180. l. 8. *f.* $\frac{1}{2}\sqrt{\frac{4x+a}{a}}$ *r.* $\frac{1}{2}\sqrt{\frac{4x+a}{x}}$. 181. l. 20. *f.* $\frac{1}{2}(a-y)$ *r.* $(\frac{1}{2}a-y)$. 182. l. 1. *f.* $\times\dot{x}^2$ *r.* $\dot{z}^2$. 197. l. 12. *f.* $y\dot{y}$ *r.* $2y\dot{y}$. 198. l. 12. *f.* x *r.* v. 200. l. 21. *f.* $\frac{1}{2}$ *r.* 2. l. 22. *f.* $\frac{1}{2}$ *r.* 2. 201. l. 1. *f.* $\frac{1}{2}v^2\dot{v}$ *r.* $2v^2\dot{v}$. l. 2. dele 4. 203. l. 17. *f.* $x-b$ *r.* $x-a$. 205. l. 27. *f.* $-$ *r.* $=$. 209. l. 4. *f.* 167. *r.* 67. l. 12. *f.* $x\dot{x}$ *r.* $\frac{x\dot{x}}{h}$; l. 13. in den. *f.* n *r.* nh; l. 17. *f.* $x^{n-1}\dot{z}$ *r.* $z^{n-1}\dot{z}$. 214. l. 14. *f.* $(e+fx)^n$ *r.* $(e+fx^n)$. 215. l. *f.* $sn+m$ *r.* $sn+n$. 216. l. 11. *f.* C *r.* A; *f.* $(m+2)$ *r.* $(m+2n)$. 217. l. 10. *f.* x *r.* $\dot{z}$. 218. l. 15. *f.* if $b =$ *r.* $b^2 =$. 228. l. 5. *f.* $ay\dot{z}$ *r.* $ay\dot{x}$. 230. l. 26. *f.* $30a^2pr$ *r.* $30a^2qr$; l. 30. *f.* $30a^2pr$ *r.* $30a^2qr$. 231. l. 2. *f.* $3a^2$ *r.* $30a^2$; l. 7. *f.* ny^{n-1} *r.* $ny^{n-1}\dot{y}$. 235. l. 8. *f.* $+\frac{x^{n-1}}{n-1}\Big) + \&c. - v$ *r.* $\frac{x^{n-1}}{n-1} + \&c. - v\Big)$. 241. l. 3. *f.* $\dot{w}$ *r.* $\dot{v}$. 251. l. 3. *f.* sign *r.* sine. 253. l. 6. *f.* y *r.* y^n. 255. l. 2. *f.* p^m *r.* p^n; l. 14. *f.* byv *r.* $by\dot{v}$. 256. l. 1. *f.* $4nx$ *r.* $4n^2x$; l. 4. *f.* nx *r.* nx^2; l. 4. *f.* pv *r.* mv; l. 5. *f.* qv *r.* pv. 257. l. 7. *f.* $1+v$ *r.* $1-v$; l. 16. *f.* $+av$ *r.* $-av$. 258. l. 13. *f.* $1+v$ *r.* $1-v$. 259. l. 10. *f.* $p-n$ *r.* $r-n$. 270. l. 14. *f.* $\sqrt{r}$ *r.* $\sqrt{a}$. 289. l. 1. *f.* horiz. *r.* cO. 294. l. 8. *f.* $-2nx$ *r.* $2nx+p$.

297. l. 21. *f.* 81 *r.* 82. **307**. l. 6. *f.* a^{m+1} *r.* a^{n+1}; l. 8. *f.* $m+1$ *r.* $n+1$. **310**. l. 11. *f.* $\sqrt{\frac{a}{d}} \times dr.$ $\sqrt{\frac{a}{d}} \times a$; l. 13. *f.* Ch. 2. *r.* Sec. 3. **317**. l. 4. *f.* $2PO$ *r.* $2BO$. **321**. in fig. *f.* L *r.* Q. **323**. l. 22. *f.* $\frac{B}{V} \times \frac{abx}{a-b}$ *r.* $\frac{B}{V} \times \frac{abx}{b-a}$. **328**. l. 16. *f.* $p\sqrt{\frac{4AD}{m}}$ *r.* $\sqrt{\frac{4AD}{m}}$. **330**. l. 13. *f.* Z *r.* z. **331**. l. 17. *f.* $1-\frac{a}{x}$ *r.* $p \times \left(1-\frac{a}{x}\right)$; *f.* $1-\frac{HO}{HA}$ *r.* $p \times \left(1-\frac{HO}{HA}\right)$. **333**. l. 16. *f.* $c^2 - \frac{c^2 P}{P+Q} . a^2$ *r.* $\frac{C^2 P}{P+Q} . a^2$. **335**. l. 11. dele *r.* **353**. l. 1. *f.* $\frac{-\sqrt{r}}{\sqrt{ms}}$ *r.* $\frac{-\sqrt{r}}{\sqrt{ms}} \times \frac{1}{\sqrt{a}}$. **355**. l. 14. *f.* $\sqrt{a+m}+\sqrt{a}$ *r.* $\sqrt{a+m}-\sqrt{a}$. **363**. l. 7. *f.* $\frac{\ddot{y}}{\dot{z}^3}$ *r.* $\frac{\ddot{y}}{\dot{z}^2}$. **365**. l. 14. *f.* $a \sim x$ *r.* $a \sim y$; *f.* bty *r.* $b+x$; *f.* $(a \sim x)^2$ *r.* $(a \sim y)^2$; l. 15. *f.* $(b+y)^2$ *r.* $(b+x)^2$; l. 17. *f.* $(a \sim x)^2 + (b+y)^2$ *r.* $(a \sim y)^2 + (b+x)^2$. **366**. l. 25 *f.* $x + AF$ *r.* $x \times AF$. **369**. l. 6. *f.* 82 *r.* 83. **371**. l. 17. *f.* $+bx$ *r.* $-bx$; changing the signs accordingly, the operation proceeds in the same manner. **374**. l. 17. *f.* $\frac{3a^3v^7}{7}$ *r.* $\frac{3a^2v^7}{7}$. **379**. l. 11. *f.* C *r.* S; l. 14. *f.* $\frac{x^2\sqrt{z}\dot{x}}{\sqrt{a}}$ *r.* $\frac{\sqrt{z}\dot{x}}{\sqrt{a}}$; l. 15. *f.* $\frac{\sqrt{z}\dot{x}}{\sqrt{a}}$ *r.* $\frac{\sqrt{z}\dot{x}}{x^2\sqrt{a}}$; l. 17. *f.* C *r.* S. **380**. l. 6. *f.* C *r.* S; l. 7. *f.* C *r.* S. **381**. l. 15. *f.* $\frac{1}{s+a}$ *r.* $\frac{a}{s+a}$; l. 24. *f.* $Sx=\dot{T}$ *r.* $S\dot{x}=\dot{T}$. **382**. l. 15. *f.* $(d-2y^2)^{m-1}$ *r.* $(d-y^2)^{m-1}$; l. 15, 16. *f.* $2A$ *r.* $2mA$; l. 16. *f.* dy^{m-1} *r.* $d^{m-1}y$. *f.* 1.3.5 *r.* 1.2.5. **383**. l. 2. *f.* 1.3.5 *r.* 1.2.5, and in the next four lines *f.* $2A$ *r.* $2mA$; l. 15. *f.* $\frac{\dot{x}^2\bar{t}}{\dot{t}^3}$ *r.* $\frac{2\dot{x}^2\bar{t}}{\dot{t}^3}$. **385**. l. 6. *f.* 2.3.4.1.6.7 *r.* 2.3.4.5.6.7. l. 7. *f.* zy *r.* $2y$. **393**. l. 12. *f.* vw *r.* va. **263**. l. 4. *f.* $=$ *r.* $+$.

The Author's absence from Cambridge must apologize for these typographical Errata.